STUDIES IN MULTIPLICATIVE NUMBER THEORY

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ABSTRACT

This thesis gives some order estimates and asymptotic formulae associated with general classes of non-negative multiplicative functions as well as some particular multiplicative functions such as the divisor functions $d_k(n)$.

In Chapter One we give a lower estimate for the number of distinct values assumed by the divisor function $d(n)$ in $1 \leqslant n \leqslant x$. We also identify the smallest positive integer which is a product of triangular numbers and not equal to $d_3(n)$ for $1 \leqslant n \leqslant x$.

In Chapter Two we show that if $f(n)$ satisfies some conditions and if

$$M = \max_{a \geqslant 1} \left\{ f(2^a) \right\}^{1/a} ,$$

then the maximum value of $f(n)$ in $1 \leqslant n \leqslant x$ is about

$$M^{\frac{\log x}{\log\log x}} .$$

We also show that a function which has a finite mean value cannot be large too often.

In Chapter Three we give an upper estimate to the average value of $f(n)$ as n runs through a short interval in an arithmetic progression with a large modulus. As an application of our general theorem we show, for example, that if $f(n)$ is the characteristic function of the set of integers which are the sum of two squares, then as $x \to \infty$,

$$\sum_{\substack{x-y<n\leq x \\ n\equiv a(\mathrm{mod}\ k)}} f(n) \ll \frac{1}{\phi(k)} \prod_{\substack{p|k \\ p\equiv 1(\mathrm{mod}\ 4)}} \left(1 - \frac{1}{p}\right) \frac{y}{\sqrt{\log x}}$$

uniformly in a, k and y provided that

$$0 < a < k\ , \quad (a,k) = 1\ , \quad k < y^{1-\alpha}\ , \quad x^{\beta} < y \leq x\ .$$

where α, β are positive constants.

We call a positive integer n a k-full integer if p^k divides n whenever p is a prime divisor of n, and in Chapter Four we give an asymptotic formula for the number of k-full integers not exceeding x. In Chapter Five we give an asymptotic formula for the number of 2-full integers in an interval. We also study the problem of the distribution of the perfect squares among the sequence of 2-full integers.

The materials in the first three chapters have been accepted for publications and will appear in [31], [22], [33] and [32].

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Chapter One

On some Results of Erdös and Mirsky

1.1 INTRODUCTION.

Throughout this chapter $c_1, c_2, \ldots$ denote absolute positive constants and p_n denotes the nth prime.

Let $d(n)$ denote, as usual, the number of positive integer divisors of n. We partition the set of all positive integers into equivalence classes $H(k)$ where

$$H(k) = \{n; d(n) = k\}, \quad k = 1,2,\ldots .$$

We call those integers which are least in their own equivalence classes the D-numbers and we denote by D the set of all D-numbers. These D-numbers form a set of representatives for the equivalence classes and the divisor function maps D bijectively onto the set of all positive integers. Since $m \in D$ implies $d(n) \neq d(m)$ for $0 < n < m$, it is clear that if we put

$$D(x) = |\{m; m \in D, m \leq x\}|$$

then $D(x)$ is the number of distinct values assumed by $d(n)$ in $1 \leq n \leq x$.

An integer m is called highly composite if $d(n) < d(m)$ for $0 < n < m$. We see that our set D contains all the highly composite integers.

In order to study $D(x)$, Erdös and Mirsky introduced in [6] another set of representatives from the equivalence classes $H(k)$, namely the set B of B-numbers, defined as integers having the form

$$p_1^{q_1-1} \; p_2^{q_2-1} \cdots p_k^{q_k-1}$$

where q_i is prime, $q_1 \geq q_2 \geq \cdots \geq q_k$ and k is arbitrary. It is easy to see that

$$p_1^{q_1-1} \cdots p_k^{q_k-1}$$

is the unique B-number in the equivalence class $H(q_1 q_2 \cdots q_k)$. Putting

$$B(x) = |\{m;\ m \in B,\ m \leq x\}|,$$

Erdös and Mirsky proved in [6], among other things, that, as $x \to \infty$,

$$\log B(x) \sim \frac{2\pi\sqrt{2}\ (\log x)^{\frac{1}{2}}}{\sqrt{3}\ \mathrm{loglog}\ x}, \tag{1.1}$$

and deduced that, as $x \to \infty$,

$$\log D(x) \sim \frac{2\pi\sqrt{2}\ (\log x)^{\frac{1}{2}}}{\sqrt{3}\ \mathrm{loglog}\ x}. \tag{1.2}$$

They also proved that there is an absolute positive constant c_1 such that, for all sufficiently large values of x,

$$D(x) - B(x) > c_1\ \mathrm{logloglog}\ x, \tag{1.3}$$

and that

$$\frac{D(x)}{B(x)} = 1 + O\left\{\frac{(\mathrm{loglog}\ x)^2}{(\log x)^{1/3}}\right\}. \tag{1.4}$$

There is a natural bijection from D to B via the equivalence classes and, because of the minimal property of D-numbers, it follows that $D(x) - B(x)$ counts precisely those D-numbers not exceeding x whose corresponding B-numbers exceed x. Unless the error term in (1.4) is exceedingly poor, (1.1) and (1.2) suggest that something better than (1.3) is true. In [31] I established the following improvement:

Let $0 < c_2 < 3^{4/3} \log 2$. Then

$$D(x) - B(x) > \exp\left\{\frac{c_2 \,(\operatorname{loglog} x)^{1/3}}{\operatorname{logloglog} x}\right\} \tag{1.5}$$

for all sufficiently large values of x. Later Dr. M. Nair and I elaborated on my method and proved:

<u>Theorem 1.</u> <u>There exists</u> $c_3 > 0$ <u>such that</u>

$$D(x) - B(x) > \exp\left\{\frac{c_3 \,(\log x)^{1/2}}{\operatorname{loglog} x}\right\},$$

<u>for all sufficiently large values of x.</u>

Compared with the upper bound for D(x) implied by (1.2) we see that, apart from the value of c_3, Theorem 1 gives the best possible result. We shall give the proof of (1.5) in section 1.2 and the proof of Theorem 1 in section 1.3. I would like to thank Dr. Nair for allowing me to include our joint work in my thesis.

In [6] Erdös and Mirsky also proved that, for $x \geq 6$, the least positive integer missed by d(n) in $1 \leq n \leq x$, that is, the least integer in the set

$$M(x) = \{m : d(n) \neq m, \quad 1 \leq n \leq x\},$$

is the smallest prime p satisfying $2^{p-1} > x$. Here we give the generalisation of the missing value problem to $d_3(n)$, the number of ways of writing n as a product of three factors. The function $d_3(n)$ is multiplicative and, for each prime p, we have

$$d_3(p^{a-1}) = \frac{1}{2}a(a+1), \qquad a = 1,2,\dots .$$

It follows that the range of $d_3(n)$ is the set R_3 of numbers which are products of triangular numbers. We now put, for $x \geq 1$,

$$M_3(x) = \left\{ m : m \in R_3,\ d_3(n) \neq m, \quad 1 \leq n \leq x \right\},$$

and we shall identify the least integer in $M_3(x)$. The generalisation is non-trivial because there is no unique factorisation of numbers of R_3 into triangular numbers. A number $m \in R_3$ is said to be reducible, or irreducible, according to whether m is, or is not, a product of smaller numbers in R_3. Every irreducible number must, of course, be triangular, but, as we shall see, there are some triangular numbers which are reducible. Even without unique factorisation these irreducible numbers do take the role of the primes in our missing value problem for $d_3(n)$ and in fact we have

Theorem 2. For $x \geq 1$ the least integer in $M_3(x)$ is the smallest irreducible number $\frac{1}{2}a(a+1)$ such that $2^{a-1} > x$.

We shall give the proof of this in section 1.4.

1.2 A LOWER BOUND FOR D(x) - B(x).

We first prove the following simple lemma.

Lemma 1. Let $1 < \alpha < 3^{4/3}$, and for each sufficiently large value of x, define r and t by

$$r = \left[\frac{\alpha(\log\log x)^{1/3}}{\log\log\log x} \right] \tag{2.1}$$

$$p_1 \dots p_t \leq x < p_1 \dots p_t p_{t+1} . \tag{2.2}$$

Then we have

$$p_1^{p_1^2} \cdots p_r^{p_r^2} < p_t \ . \tag{2.3}$$

Proof. We require the following consequences of the prime number theorem:

As $x \to \infty$,

$$\sum_{p \leq x} \log p \sim x \ , \tag{2.4}$$

$$\sum_{p \leq x} p^2 \log p \sim \frac{1}{3} x^3 \tag{2.5}$$

and

$$p_r \sim r \log r \ . \tag{2.6}$$

From (2.1) we have

$$(r \log r)^3 < (\alpha/3)^3 \text{ loglog } x \ . \tag{2.7}$$

Choosing α_1 and α_2 so that $\frac{1}{3} < \alpha_1 < \alpha_2 < (3/\alpha)^3$, we derive at once from (2.5), (2.6) and (2.7), that

$$\sum_{p \leq p_r} p^2 \log p < \alpha_1 p_r^3 < \alpha_2 (r \log r)^3$$

$$< \alpha_2 (\alpha/3)^3 \text{loglog } x = \delta \text{loglog } x \ ,$$

say, where $\delta < 1$. Therefore

$$p_1^{p_1^2} \cdots p_r^{p_r^2} = \exp\left(\sum_{p \leq p_r} p^2 \log p\right) < (\log x)^\delta . \tag{2.8}$$

Now, in view of (2.4), it follows easily from (2.2) that $p_t > \frac{1}{4} \log x$, and the desired inequality (2.3) now follows from this and (2.8).

We can now prove (1.5). Given c_2, choose α so that it satisfies

$$c_2 < \alpha \log 2 < 3^{4/3} \log 2 , \qquad (2.9)$$

and let r and t be as defined by (2.1) and (2.2). The idea ia to construct 2^{r-1} equivalence classes whose D-number representatives are less than or equal to x while the corresponding B-numbers exceed x. Let

$$u = (3, u_2, u_3, \ldots u_r)$$

be a vector with each component $u_i (2 \leq i \leq r)$ either 1 or 2. To each such vector u we assign the equivalence class

$$H_u = H\left(2^{t-r+2} p_2^{u_2} p_3^{u_3} \ldots p_r^{u_r}\right) ,$$

and let d_u and b_u be respectively the D-number and B-number in H_u. We note immediately that b_u has exactly $t - r + 2 + u_2 + u_3 + \ldots + u_r$ distinct prime factors. Since each $u_i \geq 1$, we have, by (2.2), that

$$b_u \geq p_1 \cdots p_{t-r+2+r-1} = p_1 \cdots p_{t+1} > x .$$

Next, note that in order to prove $d_u \leq x$, it suffices to construct a number $a_u \in H_u$ such that $a_u \leq x$. Now a suitable candidate is the number

$$a_u = 2^7 p_2^{w_2} p_3^{w_3} \cdots p_r^{w_r} p_{r+1} \cdots p_{t-1} ,$$

where w_i denotes $p_i^{u_i} - 1$, for it is easy to verify that $a_u \in H_u$ by computing $d(a_u)$. Moreover, since each $u_i \leq 2$ we have, by (2.3) and (2.2), that

$$a_u \leq p_2^{p_2^2} p_3^{p_3^2} \cdots p_r^{p_r^2} p_{r+1} \cdots p_{t-1}$$

$$< p_{r+1} \cdots p_{t-1} p_t < x .$$

We have therefore proved that $d_u < x < b_u$. Since the number of vectors u considered is 2^{r-1}, it now follows from (2.1) and (2.9) that

$$D(x) - B(x) \geq 2^{2-1} = \exp\left\{(r-1)\log 2\right\}$$

$$\geq \exp\left\{\log 2\left(\frac{\alpha(\log\log x)^{1/3}}{\log\log\log x} - 2\right)\right\}$$

$$> \exp\left\{\frac{c_2(\log\log x)^{1/3}}{\log\log\log x}\right\},$$

and (1.5) is proved.

1.3 PROOF OF THEOREM 1.

The idea in section 1.2 is to construct 2^{r-1} equivalence classes, with r as large as the method allows, such that in each equivalence class the D-number does not exceed x, whereas the B-number exceeds x. Accordingly we now set

$$r = \left[\frac{c_4(\log x)^{1/2}}{\log\log x}\right]. \tag{3.1}$$

We consider the vector

$$u = (u_1, u_2, \ldots, u_r)$$

where each component u_i $(2 \leq i \leq r)$ is either 0 or 1. The component u_1, which will be specified later, depends on the individual choices of u_i $(2 \leq i \leq r)$, so that the total number of such vectors we consider is 2^{r-1}. To each such vector u we assign the equivalence class

$$H_u = H\left(p_1^{u_1}\, p_2^{u_2} \ldots p_r^{u_r}\right).$$

Let d_u and b_u be the D-number and B-number in H_u, and our task is to prove that

$$d_u \leqslant x < b_u \ . \tag{3.2}$$

From the definition of a B-number, we must have

$$b_u = g_r^{p_r-1} \ g_{r-1}^{p_{r-1}-1} \cdots g_2^{p_2-1} \ g_1 \tag{3.3}$$

where

$$g_i = p_{u_r+u_{r-1}+\ldots+u_{i+1}+1} \cdots p_{u_r+u_{r-1}+\ldots+u_i}, \qquad 1 \leqslant i \leqslant r,$$

with the convention that an empty sum is 0 and an empty product is 1. Note that each g_i has exactly u_i prime factors so that

$$d(g_i^{p_i-1}) = p_i^{u_i}$$

and whence $\qquad b_u \in H_u.$

Let us write

$$G = g_r^{p_r-1} \ g_{r-1}^{p_{r-1}-1} \ldots . \ g_2^{p_2-1} \tag{3.4}$$

and we note that

$$G \leqslant (p_{r-1}^{\ r-1})^{p_{r-1}} < \exp\ (r\ p_r \log p_r)$$

$$< \exp\ (2r^2 \log^2 r) < x^{1/2}$$

by (3.1). Moreover, since

$$p_{u_r+u_{r-1}+\ldots+u_2+1} \leqslant p_r < (\log x)^{1/2} ,$$

given any choice for $(u_2, u_3, \ldots, u_r)$ we can define u_1 in terms of this choice by

$$G\, p_{u_r+\dots+u_2+1} \cdots p_{u_r+\dots+u_1-1} \leqslant x < G\, p_{u_r+\dots+u_2+1} \cdots p_{u_r+\dots+u_1} .$$

(3.5)

Indeed it is easy to show that u_1 satisfies

$$\frac{\log x}{10 \operatorname{loglog} x} < u_1 < \frac{10 \log x}{\operatorname{loglog} x} \qquad (3.6)$$

uniformly with respect to $(u_2, u_3, \dots, u_r)$.

From (3.3), (3.4) and (3.5) it follows at once that $b_u > x$. It remains to show that $d_u \leqslant x$, and, because of the minimal property of a D-number, it suffices to construct a number $a_u \leqslant x$ such that $a_u \in H_u$. Now a suitable candidate is the number

$$a_u = G(p_{u_r+\dots+u_2+1} \cdots p_{u_r+\dots+u_2+r})^3 (p_{u_r+\dots+u_2+r+1} \cdots p_{u_r+\dots+u_2+r+\ell})$$

where ℓ satisfies $u_1 = 2r + \ell$. We note that, from (3.1) and (3.6),

$$\ell \sim u_1, \text{ as } x \to \infty . \qquad (3.7)$$

Now

$$d(a_u) = d(G)\, 4^r\, 2^\ell = d(G)\, 2^{2r+\ell}$$

$$= p_r^{u_r}\, p_{r-1}^{u_{r-1}} \cdots p_2^{u_2}\, p_1^{u_1}$$

so that $a_u \in H$. Moreover, from (3.5) we see that in order to prove $a_u \leqslant x$ it suffices to show that

$$(p_{u_r+\dots+u_2+1} \cdots p_{u_r+\dots+u_2+r})^2 \leqslant p_{u_r+\dots+u_r+r+\ell+1} \cdots p_{u_r+\dots+u_2+2r+\ell-1} .$$

(3.8)

Now the left hand side of (3.8) is at most

$$p_{2r}^{\;2r} \leqslant \exp(2r \log r + 2r \operatorname{loglog} r + O(r))$$

whereas the right hand side of (3.8) is at least

$$p_{\ell}^{\;r-1} \geqslant \exp(r \log p_{\ell} + O(\log p_{\ell}))$$

$$\geqslant \exp(r \log \ell + r \operatorname{loglog} \ell + O(r) + O(\log p_{\ell}))$$

$$\geqslant \exp(r \operatorname{loglog} x + O(r))$$

by (3.6) and (3.7). Thus (3.8) follows if

$$2 \log r + 2 \operatorname{loglog} r + O(1) \leqslant \operatorname{loglog} x,$$

and this certainly holds provided that the constant c_4 in (3.1) is sufficiently small to absorb the bounded term. Therefore (3.2) is proved, and so

$$D(x) - B(x) \geqslant 2^{r-1} = \exp\left\{(r-1) \log 2\right\},$$

and the theorem follows from (3.1) and this.

1.4 PROOF OF THEOREM 2.

It is convenient to write

$$\overline{a} = \frac{1}{2} a(a+1), \qquad a = 1,2,\ldots .$$

Examples of reducible triangular numbers are:

$$\overline{8} = \overline{3}.\overline{3}, \qquad \overline{9} = \overline{2}.\overline{5}, \qquad \overline{20} = \overline{4}.\overline{6} .$$

An example of non-unique factorisation in R_3 is:

$$\overline{2}.\overline{14} = \overline{5}.\overline{6} .$$

Indeed, even a triangular number itself may have two different sets of irreducible factors:

$$\overline{35} = \overline{3}.\overline{14} = \overline{2}.\overline{4}.\overline{6} \quad .$$

Let $\overline{a}$ and $\overline{b}$ be two successive irreducible numbers. There are b-a-1 reducible triangular numbers between them and we conjecture that this gap is bounded above by an absolute constant. However, the upper bound a/2 is quite sufficient for our purpose, and we now prove the following.

<u>Lemma 1.</u> <u>Let</u> $6 \leqslant \overline{a} < \overline{b}$. <u>If</u> $\overline{a}$ <u>and</u> $\overline{b}$ <u>are successive irreducible numbers, then</u>

$$b-a < \frac{a}{2} \quad \text{or} \quad 2b < 3a.$$

<u>Proof.</u> For $3 \leqslant a \leqslant 21$, the lemma can be verified by computing b. Assume therefore that $a \geqslant 22$. Now by a strong version of Bertrand's postulate, there is a prime p such that

$$a < p-1 < \frac{3}{2}\, a.$$

Moreover the triangular number $\overline{p-1} = \frac{1}{2}\,(p-1)p$ is the smallest triangular number divisible by p and is therefore necessarily irreducible. Since $\overline{b}$ is the irreducible number immediately succeeding $\overline{a}$ we now have

$$a < b \leqslant p-1 < \frac{3}{2}\, a$$

and the required result follows.

Theorem 2 can be verified by direct computation if $x \leqslant 3072$. We therefore assume that $x > 3072$.

Firstly, if $2^{a-1} > x$ and $\bar{a}$ is an irreducible number, then certainly $\bar{a} \in M_3(x)$. Let $m < \bar{a}$ and $m \in R_3$. We have to prove that $m \notin M_3(x)$; that is we have to find a solution to

$$d_3(n) = m \qquad 1 \leqslant n \leqslant x. \tag{4.1}$$

Let us denote by $\bar{b}$ the irreducible triangular number immediately preceeding $\bar{a}$ so that

$$2^{b-1} \leqslant x. \tag{4.2}$$

Now if m itself is irreducible, then $m = \bar{a}_1$ where $a_1 \leqslant b$, and we see by (4.2) that

$$n = 2^{a_1 - 1}$$

is now a suitable solution to (4.1). We assume therefore that m is a reducible number so that

$$m = \bar{a}_1 \ldots \bar{a}_\ell, \qquad a_1 \geqslant \ldots \geqslant a_\ell$$

where $\ell \geqslant 2$. We now put

$$n = p_1^{a_1 - 1} \ldots p_\ell^{a_\ell - 1}$$

so that $d_3(n) = m$. In view of (4.2) we see that (4.1) will have a solution if $n \leqslant 2^{b-1}$; that is, if

$$(a_1 - 1) \log p_1 + \ldots + (a_\ell - 1) \log p_\ell \leqslant (b-1) \log 2.$$

It suffices therefore to prove that

$$(a_2 - 1)(\log p_2 + \ldots + \log p_\ell) \leqslant (b - a_1) \log 2,$$

that is

$$b - a_1 \geqslant \left\{\frac{\theta(p_\ell)}{\log 2} - 1\right\} (a_2 - 1) \tag{4.3}$$

where, as usual, $\theta(p_\ell) = \log p_1 + \ldots + \log p_\ell$.

Now from $\bar{a}_1 \ldots \bar{a}_\ell = m < \bar{a}$ we deduce that

$$\frac{a-1}{a_1-1} > \frac{a}{a_1} > \frac{a+1}{a_1+1}$$

whence

$$\left(\frac{a-1}{a_1-1}\right)^2 > \frac{a(a+1)}{a_1(a_1+1)} = \frac{\bar{a}}{\bar{a}_1} > \bar{a}_2 \ldots \bar{a}_\ell$$

and so

$$a - 1 > (\bar{a}_2 \ldots \bar{a}_\ell)^{1/2} (a_1 - 1).$$

By Lemma 1 we also have $3b \geq 2a + 1$ so that

$$b - a_1 \geq \frac{2}{3}(a-1) - (a_1 - 1)$$

and we arrive at

$$b - a_1 > \left\{\frac{2}{3}(\bar{a}_2 \ldots \bar{a}_\ell)^{1/2} - 1\right\}(a_1 - 1). \tag{4.4}$$

Here we note that (4.3) follows from (4.4) at once if

$$\theta(p_\ell) \leq \frac{2}{3} \log 2\ (\bar{a}_2 \ldots \bar{a}_\ell)^{1/2}. \tag{4.5}$$

Since $\bar{a}_2 \geq \ldots \geq \bar{a}_\ell \geq 3$, we see that (4.5) certainly holds if $\ell \geq 8$.

To see this we note that, for $\ell \geq 9$,

$$\theta(p_\ell) \leq (2 \log 2)\, p_\ell, \qquad p_\ell \leq 3^{\frac{\ell-3}{2}}$$

so that we have

$$\theta(p_\ell) \leq \left(\frac{2}{3} \log 2\right) 3^{\frac{\ell-1}{2}}$$

and it is easy to check that this holds even for $\ell = 8$. We are now left to deal with the cases $2 \leq \ell \leq 7$ with the additional assumption that (4.5) does not hold; that is

$$\bar{a}_2 \ldots \bar{a}_\ell < \left(\frac{3\theta(p_\ell)}{2 \log 2}\right)^2 .$$

But this implies that

$$\bar{a}_2 < 3^{4-\ell} \left(\frac{\theta(p_\ell)}{2 \log 2}\right)^2$$

and we see that $\bar{a}_2$ is bounded above by 15, 15, 10, 10, 6 and 3 for $\ell = 2, 3, 4, 5, 6$ and 7 respectively. The corresponding upper bound for a_2 itself is thus 5, 5, 4, 4, 3 and 2 respectively. Since $x > 3072$ we can take a_1 large enough so that (4.4) implies (4.3). For example, we take the worst case when $\ell = 2$ and $a_2 = 2$. Since

$$x > 3072 = 2^{11-1}\, 3^{2-1}$$

we see that if $m = \bar{a}_1 \bar{a}_2$ where $a_1 \leq 11$ and $a_2 = 2$, then certainly $m \in M_3(x)$, and therefore we can safely assume that $a_1 \geq 12$, and now the right hand side of (4.4) is

$$\left(\frac{2\sqrt{3}}{3} - 1\right)(12 - 1) > 1.7$$

whereas the right hand side of (4.3) is

$$\frac{\log 3}{\log 2} < 1.6 \quad .$$

The theorem is proved.

1.5 THE LEAST REDUCIBLE NUMBER IN $M_3(x)$

Returning to the original missing value problem in $M(x)$ considered by Erdös and Mirsky, we remark that, for large x, one would expect not just the first number, but the initial block of numbers in $M(x)$ to be primes. This is indeed the case; for it can be proved that, for $x \geq 36$, the first composite number in $M(x)$ is $2p$ where p is the least prime satisfying $3.2^{p-1} > x$. The problem of identifying the first pair of consecutive numbers in $M(x)$ seems very difficult. We conjecture that, for large x, they are numbers of the form $2p-1$, $2p$ or $2p$, $2p+1$ where in either case, p, $2p\pm1$ are primes. It is not known if there are infinitely many primes of the form $2p\pm1$.

One might guess that the least reducible number in $M_3(x)$ is $3\bar{a}$ where $\bar{a}$ is the smallest irreducible number satisfying $3.2^{a-1} > x$; but this is not always the case. We recall that there is no unique factorisation in R_3, and therefore if $\bar{a}$ is an irreducible number, $3\bar{a}$ may still be factorised into other triangular numbers. If this is so, then $3\bar{a}$ may not be a missing value after all. For example, $\overline{14}$ is an irreducible number but

$$3.\overline{14} = \bar{5}.\bar{6}$$

and so the least solution to $d_3(n) = 3.\overline{14}$ is actually $2^5.3^4$ and not $2^{13}.3$. However we do have the following:

<u>Theorem 3.</u> <u>For</u> $x \geq 1$, <u>the least reducible number in</u> $M_3(x)$ <u>is</u> <u>the number</u> $3\bar{a}$ <u>where</u> a <u>is the smallest integer satisfying</u>

(i) $3.2^{a-1} > x$

(ii) $3\bar{a}$ has no other factorisation in R_3.

We shall omit the proof of this theorem since it is very similar to the proof of Theorem 2.

Chapter Two

The Maximum Orders of Multiplicative Functions

2.1 INTRODUCTION.

Let $d(n)$ be the divisor function. It is well known (see, for example, Hardy and Wright [11], Theorem 317, p.262) that

$$\limsup_{n \to \infty} \frac{\log d(n) \log\log n}{\log n} = \log 2. \tag{1.1}$$

Let $\alpha(n)$ denote the number of non-isomorphic Abelian groups of order n. Kendall and Rankin [14] proved that

$$\frac{1}{2} \log 2 \leqslant \limsup_{n \to \infty} \frac{\log \alpha(n) \log\log n}{\log n} \leqslant \frac{2\pi}{\sqrt{6}} ,$$

and later Krätzel [16] proved that actually

$$\limsup_{n \to \infty} \frac{\log \alpha(n) \log\log n}{\log n} = \frac{1}{4} \log 5. \tag{1.2}$$

We call a positive integer m square-full if m is a product of squares and cubes. Let $\beta(n)$ denote the number of square-full divisors of n. Knopfmacher [15] proved that

$$\limsup_{n \to \infty} \frac{\log \beta(n) \log\log n}{\log n} = \frac{1}{3} \log 3. \tag{1.3}$$

The arguments used in the proofs of (1.2) and (1.3) are parallel to that of (1.1) given in Hardy and Wright, and the main purpose of this chapter is to show that the method is applicable to a general class of multiplicative functions. We shall prove:

Theorem 1. Let $f(n)$ be a multiplicative function satisfying the following conditions.

(i) There exist constants A and θ $(0 < \theta < 1)$ such that

$$f(2^a) \leqslant \exp(Aa^\theta), \qquad a \geqslant 1.$$

(ii) For all primes p, and all $a \geqslant 1$,

$$f(p^a) = f(2^a) \geqslant 1.$$

Then we have

$$\limsup_{n \to \infty} \frac{\log f(n) \operatorname{loglog} n}{\log n} = \log M$$

where

$$M = \max_{a \geqslant 1} (f(2^a))^{1/a}.$$

It is also well known (Hardy and Wright [11], Theorem 432, p.359) that if $\varepsilon > 0$, then

$$|\log d(n) - \log 2 \operatorname{loglog} n| < \varepsilon \operatorname{loglog} n \tag{1.4}$$

for almost all n. In [14] and [15] respectively, it is proved that

$$\log \alpha(n) < \left(\frac{2\pi}{\sqrt{6}} + \varepsilon\right) \operatorname{loglog} n, \tag{1.5}$$

and

$$\log \beta(n) < \left(\frac{1}{3} \log 3 + \varepsilon\right) \operatorname{loglog} n, \tag{1.6}$$

for almost all n. Actually (1.5) and (1.6) are rather misleading in that they have the following drastic improvement: given any unbounded increasing function $g(n)$, we have $\alpha(n) < g(n)$ and $\beta(n) < g(n)$ for almost all n. This follows from the following simple theorem.

Theorem 2. *Let $f(n)$ be any non-negative function satisfying the condition $\sum_{n \leq x} f(n) = O(x)$ as $x \to \infty$. Then, for any unbounded increasing function $g(n)$, we have $f(n) < g(n)$ for almost all n.*

2.2 PROOF OF THEOREM 1.

That M exists follows from

$$1 \leq (f(2^a))^{1/a} \leq \exp(Aa^{\theta-1}) \to 1 \text{ as } a \to \infty.$$

We first show that

$$\limsup_{n \to \infty} \frac{\log f(n) \operatorname{loglog} n}{\log n} \geq \log M . \tag{2.1}$$

We can choose b so that $M^b = f(2^b)$. Let $p_1, p_2, \ldots, p_r$ be the first r primes and set

$$n = (p_1 p_2 \cdots p_r)^b$$

so that

$$f(n) = (f(2^b))^r = M^{br} = M^{b\pi(p_r)} .$$

From the prime number theorem we have, as $r \to \infty$,

$$\pi(p_r) \sim \frac{p_r}{\log p_r}, \quad \sum_{p \leq p_r} \log p \sim p_r ,$$

so that

$$\log n = b \sum_{p \le p_r} \log p \sim bp_r ,$$

and

$$\log\log n = \log p_r + O(1).$$

Consequently we have, as $r \to \infty$, that

$$\log f(n) = b \log M \,.\, \pi(p_r)$$

$$\sim \frac{b \log M \,.\, p_r}{\log p_r} \sim \frac{\log M \log n}{\log\log n}$$

and so (2.1) is proved.

In order to prove that

$$\limsup_{n \to \infty} \frac{\log f(n) \log\log n}{\log n} \leqslant \log M \tag{2.2}$$

we shall require the following lemma.

<u>Lemma.</u> <u>Let</u> $A > 0$ <u>and</u> $0 < \theta < 1$. <u>Then there exist constants</u> B <u>and</u> λ <u>such that</u>

$$\frac{B}{\delta^{\lambda}} + a\ \delta \log 2 \geqslant Aa^{\theta}$$

<u>for all</u> $a \geqslant 1$ <u>and all</u> $\delta > 0$.

<u>Proof.</u> We set

$$\lambda = \frac{\theta}{1-\theta} , \qquad B = \left(\frac{A\theta}{\lambda}\right)^{\frac{\lambda}{\theta}} \left(\frac{\lambda}{\log 2}\right)^{\lambda} ,$$

$$x = \frac{B}{\delta^{\lambda}} , \qquad y = \frac{a\ \delta \log 2}{\lambda} .$$

The required result follows at once from the inequality $(1-\theta)x + \theta y \geqslant x^{1-\theta}y^{\theta}$.

Returning to the proof of (2.2), let $n = \Pi p^a$ be the factorisation of n into prime powers, and $\delta > 0$. We have

$$\frac{f(n)}{n^\delta} = \Pi \frac{f(p^a)}{p^{a\delta}} = \Pi \frac{f(2^a)}{p^{a\delta}} \quad . \tag{2.3}$$

From condition (i) we have, for all p and a, that

$$\frac{f(p^a)}{p^{a\delta}} \leqslant \frac{f(2^a)}{p^{a\delta}} \leqslant \exp(Aa^\theta - a\,\delta \log 2) \leqslant \exp\left(\frac{B}{\delta^\lambda}\right)$$

by our lemma. For $p \geqslant M^{1/\delta}$ we also have that

$$\frac{f(2^a)}{p^{a\delta}} \leqslant \frac{f(2^a)}{M^a} \leqslant 1.$$

It now follows from (2.3) that

$$\frac{f(n)}{n^\delta} \leqslant \prod_{p \leqslant M^{1/\delta}} \exp\left(\frac{B}{\delta^\lambda}\right) \leqslant \exp\left(\frac{BM^{1/\delta}}{\delta^\lambda}\right)$$

so that

$$\log f(n) \leqslant \delta \log n + \frac{BM^{1/\delta}}{\delta^\lambda} \quad .$$

Now let $\varepsilon > 0$ and set

$$\delta = \frac{\left(1 + \frac{\varepsilon}{2}\right) \log M}{\log\log n}$$

so that

$$\log f(n) \leqslant \frac{\left(1 + \frac{\varepsilon}{2}\right) \log M \log n}{\log\log n} + \frac{B(\log n)^{\frac{1}{1 + \frac{\varepsilon}{2}}} (\log\log n)^\lambda}{\left(1 + \frac{\varepsilon}{2}\right)^\lambda \log^\lambda M}$$

$$\leqslant \frac{(1 + \varepsilon) \log M \log n}{\log\log n} \quad ,$$

provided that n is sufficiently large. This proves (2.2) and so completes the proof of Theorem 1.

2.3 PROOF OF THEOREM 2.

Let $S = \{n : f(n) \geqslant g(n)\}$ and suppose, if possible, that S has positive upper asymptotic density δ. Then there exists arbitrarily large x such that

$$\sum_{n \leqslant x} f(n) \geqslant \sum_{\substack{n \leqslant x \\ n \in S}} g(n) \geqslant \sum_{n \leqslant \frac{1}{2}\delta x} g(n) ,$$

since there must be at least $\frac{1}{2}\delta x$ positive integers belonging to S in the interval $1 \leqslant n \leqslant x$, and $g(n)$ is an increasing function. We now have that

$$\sum_{n \leqslant x} f(n) \geqslant \sum_{\frac{1}{4}\delta x \leqslant n \leqslant \frac{1}{2}\delta x} g(n) \geqslant \tfrac{1}{4}\delta x g\left(\left[\frac{\delta x}{4}\right]\right) .$$

Since $g(n)$ is also unbounded, this contradicts $\sum_{n \leqslant x} f(n) = O(x)$, as $x \to \infty$. Therefore S must have zero asymptotic density and the theorem is proved.

2.4 REMARKS.

1. We have $d(2^a) = a + 1$, $\beta(2^a) = a$ and $\alpha(2^a) = P(a)$ where $P(a)$ is the number of partitions of a into positive parts. It is well known (see, for example, Apostol [1], Theorem 14.7, p.316) that $P(a) < \exp(A\sqrt{a})$ with $A = 2\pi/\sqrt{6}$, so that our Theorem 1 is applicable with $\theta = \frac{1}{2}$. We see that (1.1), (1.2) and (1.3) follow from the fact that

$$(a+1)^{1/a}\ ,\ (P(a))^{1/a}\ \text{ and } a^{1/a}$$

have maximum values at a = 1,4 and 3 respectively.

2. It is clear from the proof of Theorem 1 that if f(n) satisfies only the condition

$$f(p^a) \leqslant \exp(Aa^\theta),\ p \text{ prime},\quad a \geqslant 1,$$

then we can still deduce that

$$\limsup_{n\to\infty} \frac{\log f(n)\ \log\log n}{\log n} \leqslant \log M^*$$

where

$$M^* = \sup_p \left\{\max_{a\geqslant 1} (f(p^a))^{1/a}\right\}.$$

3. It is shown in [14] and [15] respectively that

$$\sum_{n\leqslant x} \alpha(n) \sim \prod_{r=2}^{\infty} \zeta(r)\ .\ x = (2.29\ldots)x, \quad \text{as } x\to\infty\ ,$$

and

$$\sum_{n\leqslant x} \beta(n) \sim \frac{\zeta(2)\zeta(3)}{\zeta(6)}\ x = (1.94\ldots)x, \quad \text{as } x\to\infty\ ,$$

whereas

$$\sum_{n\leqslant x} d(n) \sim x\log x, \quad \text{as } x\to\infty\ ,$$

so that $\alpha(n)$ and $\beta(n)$ cannot be compared with $d(n)$ in relation to (1.4). We saw that, for $a > 4$, $\alpha(p^a)$ is much larger than $d(p^a)$, but $\alpha(p) = \beta(p) = 1$ whereas $d(p) = 2$. Thus, as is to be expected, the behaviour of a multiplicative function depends most heavily on its values at the primes, and less on its values at prime powers.

Chapter Three

A Brun-Titchmarsh Theorem for Multiplicative Functions

3.1 INTRODUCTION.

Let d(n) be the divisor function, and let a and k be integers satisfying

$$0 < a < k, \qquad (a,k) = 1. \tag{1.1}$$

In 1957 Linnik and Vinogradov [20] proved that if $0 < \alpha < \frac{1}{2}$, then, as $x \to \infty$,

$$\sum_{\substack{n \leq x \\ n \equiv a (\mathrm{mod}\ k)}} d(n) << \frac{\phi(k)}{k^2} x \log x \tag{1.2}$$

uniformly in a and k, provided only that $k < x^{1-\alpha}$. Here $\phi(k)$ is Euler's function and the implied constant depends only on α. Their proof reduces to estimating the number $\psi(x,y)$ of positive integers not exceeding x having no prime factor greater than y. Unfortunately they made use of a uniform upper estimate of $\psi(x,y)$ by A.I. Vinogradov which is incorrect (see Norton [23]). Later Linnik [19] stated the generalisation of the problem to the functions

$$d_r^{\ell}(n) = (d_r(n))^{\ell}, \qquad r = 2,3,\ldots, \qquad \ell = 1.2,\ldots$$

where $d_r(n)$ is the number of ways of writing n as a product of r factors, taking account of ordering. The expected result is that, as $x \to \infty$,

$$\sum_{\substack{n \leq x \\ n \equiv a (\mathrm{mod}\ k)}} d_r^{\ell}(n) \ll \frac{x}{k} \left(\frac{\phi(k)}{k} \log x\right)^{r^{\ell}-1} \tag{1.3}$$

uniformly in $k < x^{1-\alpha}$ with the implied constant depending on r, ℓ and α. This result has been used by others (see, for example [21]) to tackle a variety of problems.

The merit of these results centres on the range of uniformity for the modulus k being large. Indeed, in the shorter range $k < x^{2/3-\alpha}$, Selberg (unpublished manuscript) has obtained even an asymptotic formula for the sum in (1.2) (see also Hooley [13] and Heath-Brown [12]). The situation is similar to that in estimating $\pi(x;k,a)$, the number of primes $p \leq x$ such that $p \equiv a(\mathrm{mod}\ k)$; the Brun-Titchmarsh inequality

$$\pi(x;k,a) \ll \frac{x}{\phi(k) \log \frac{x}{k}},$$

valid uniformly in $k < x$, is often used to supplement the asymptotic formula for $\pi(x;k,a)$ given by the Siegel-Walfisz theorem, which is known to be valid only for a much shorter range of k. Indeed, the Brun-Titchmarsh inequality is sometimes used to supplement even Bombieri's theorem; for example, in the solution to the Titchmarsh-Linnik divisor problem (see [10]).

We remark however that, unlike the Brun-Titchmarsh inequality, (1.2) cannot be extended all the way to $k < x$. To see this, we choose k to be the largest prime less than x; then the right hand side of (1.2) is of order $\log x$, whereas we can choose $a < k$ such that $d(a) > \log^2 x$.

In 1971 Wolke [39] applied the method of Erdös [5] to study the sum $\sum_{n \leq x} f(a_n)$ where f is a non-negative multiplicative function satisfying the condition that there exist positive constants c_1 and c_2 such that for all primes p and all $\ell \geq 1$,

$$f(p^{\ell}) \leq c_1 \ell^{c_2} , \tag{1.4}$$

and (a_n) is a strictly increasing sequence of positive integers. However, the results they obtained are not uniform with respect to any given class of sequences (a_n), such as the class of arithmetic progressions. Nevertheless, using their method, we can now give the generalisation of the Brun-Titchmarsh problem for a class of non-negative multiplicative functions f which satisfy conditions weaker than (1.4), and in a short interval $x - y < n \leq x$.

In section 3.6 we apply our main theorem to give a proof of (1.3), and to other results. In section 3.7 we use the method to give a new proof, and a generalisation, of a famous result of Turan, (see [37]) namely that, as $x \to \infty$,

$$\sum_{n \leq x} w^2(n) = x(\log\log x)^2 + O(x \log\log x)$$

Here w(n) denotes the number of distinct prime factors of n.

3.2 THE MAIN THEOREM.

We shall consider the class M of arithmetic functions of which are non-negative, multiplicative and satisfy the following two conditions.

(i) There exists a positive constant A_1 such that

$$f(p^{\ell}) \leqslant A_1^{\ell}, \qquad p \text{ prime}, \qquad \ell \geqslant 1.$$

(ii) For every $\varepsilon > 0$, there exists a positive constant $A_2 = A_2(\varepsilon)$ such that

$$f(n) \leqslant A_2\, n^{\varepsilon}, \qquad n \geqslant 1 .$$

The following is our main theorem.

Theorem 1. *Let* $f \in M$, $0 < \alpha < \frac{1}{2}$, $0 < \beta < \frac{1}{2}$ *and let* a, k *be integers satisfying*

$$0 < a < k, \qquad (a,k) = 1 .$$

Then, as $x \to \infty$,

$$\sum_{\substack{x-y<n\leqslant x \\ n\equiv a \pmod k}} f(n) \ll \frac{y}{\phi(k)} \frac{1}{\log x} \exp\left(\sum_{\substack{p\leqslant x \\ p\nmid k}} \frac{f(p)}{p}\right), \tag{2.1}$$

uniformly in a,k *and* y *provided that*

$$k < y^{1-\alpha}, \qquad x^{\beta} < y \leqslant x . \tag{2.2}$$

We make the following remarks on our conditions (i) and (ii). First, the condition (1.4) clearly implies (i) and we now show that it also implies (ii). Since f(n) is multiplicative we only need to consider the case $n = p$. Choose c_1, c_2 accordingly and let $\varepsilon > 0$. There exists A_2 such that, for all ℓ,

$$c_1 \ell^{c_2} \leqslant A_2 (2^\varepsilon)^\ell .$$

It follows that

$$f(n) = f(p^\ell) \leqslant c_1 \ell^{c_2} \leqslant A_2 (2^\varepsilon)^\ell$$

$$\leqslant A_2 p^{\ell\varepsilon} = A_2 n^\varepsilon .$$

We next show that there exists $f \in M$ such that (1.4) does not hold. Consider the function

$$f(n) = \begin{cases} \exp\left(\dfrac{\log n}{\log\log n}\right) & \text{if } n = 3^\ell , \quad \ell \geqslant 3 , \\ 1 & \text{otherwise.} \end{cases}$$

We see that $f(n)$ is multiplicative and

$$f(3^\ell) = \exp\left(\frac{\ell \log 3}{\log \ell + \log\log 3}\right), \qquad \ell \geqslant 3 .$$

With $A = e^5$ we have

$$f(3^\ell) \leqslant \exp(5\ell) = A^\ell .$$

Thus condition (i) is satisfied, and clearly (ii) also holds, so that $f \in M$. But, for any fixed c_1, c_2 we have

$$f(3^\ell) \geqslant \exp(\sqrt{\ell}) > c_1 \ell^{c_2}$$

for sufficiently large ℓ. Therefore (1.4) cannot hold.

Next we note that if $f \in M$ and $\delta > 0$, then there exists $A_3 = A_3(\delta)$ such that

$$\sum_p \sum_{\ell=2}^{\infty} \frac{f(p^\ell)}{p^{\ell\delta}} \leqslant A_3 . \tag{2.3}$$

Finally we point out that the rather strong condition (ii) is actually vital to the truth of the theorem, and, in particular, we cannot replace it by (2.3). For suppose that (ii) does not hold; then there exist $\varepsilon_o > 0$ and a strictly increasing sequence of positive integers (n_r) such that $f(n_r) > n_r^{\varepsilon_o}$. We can now show that (2.1) does not hold by setting $\alpha = \frac{1}{10} \min(\varepsilon_o, 1)$. We define a,k,x and y as follows: For large r, we put

$$a = n_r,$$

$$k = \text{the least prime in the interval } (a, 2a),$$

$$x = y = k^{\frac{1}{1-\alpha}} + 1.$$

Then the left hand side of (2.1) is at least $f(a) > a^{\varepsilon_o}$, whereas the right hand side is at most

$$\frac{y}{\phi(k)}\, y^{\alpha} = \frac{y^{1+\alpha}}{k-1} << k^{\frac{1+\alpha}{1-\alpha} - 1} << k^{3\alpha} << a^{3\alpha} < a^{3\varepsilon_o/10}.$$

We note, in particular, that Theorem 1 cannot be extended to cover the case when $f(n) = 2^{\Omega(n)}$, where $\Omega(n)$ denotes the total number of prime factors of n.

3.3 NOTATIONS.

The letters a, b, d, h, i, j, k, ℓ, m, n, r and s are used to denote positive integers and we shall assume that (1.1) always holds. The letters p and q are reserved for prime numbers. We let p(n) and q(n) denote the greatest and the least prime factors of n respectively. We let w(n) denote the number of distinct prime factors of n while $\Omega(n)$ is the total number of prime factors of n, taking account of multiplicity. We also define p(1), q(1), w(1), $\Omega(1)$ to be 1. As usual r(n) is the number of ways of writing n as a sum of two squares. We call a positive integer s a square full integer if s is a product of squares and cubes, and we let $\delta(n)$ be the number of square full divisors of n.

The letters x, y, z, α, β, δ and ε denote positive real numbers and λ is a real number. We put

$$\Psi(x,z) = \sum_{\substack{n\leq x \\ p(n)\leq z}} 1$$

and

$$\Phi(x,y,z;k,a) = \sum_{\substack{x-y<n\leq x \\ n\equiv a(\mathrm{mod}\ k) \\ q(n)>z}} 1 .$$

All the constants implied by the << and O-notations depend at most on α,β and f.

3.4 PRELIMINARY LEMMAS.

We shall require the following lemmas.

Lemma 1. For all sufficiently large x, we have

$$\Psi(x, \log x \operatorname{loglog} x) \leq \exp\left\{\frac{3 \log x}{(\operatorname{loglog} x)^{1/2}}\right\} .$$

Proof. As a consequence of the prime number theorem, we have, for all large y, that

$$\sum_{p\leq y} \frac{1}{\log p} \leq \frac{2y}{\log^2 y} . \tag{4.1}$$

We now use Rankin's method (see [25]). Let $\delta > 0$. We have, for large y,

$$\Psi(x,y) = \sum_{\substack{n\leq x\\ p(n)\leq y}} 1 \leq x^{\delta} \sum_{\substack{n\leq x\\ p(n)\leq y}} \frac{1}{n^{\delta}}$$

$$\leq x^{\delta} \sum_{\substack{n\geq 1\\ p(n)\leq y}} \frac{1}{n^{\delta}} = x^{\delta} \prod_{p\leq y} \left(1 + \frac{1}{p^{\delta}} + \frac{1}{p^{2\delta}} + \dots\right)$$

$$= x^{\delta} \prod_{p\leq y} \left(1 + \frac{1}{p^{\delta}-1}\right) \leq \exp\left(\delta \log x + \sum_{p\leq y} \frac{1}{p^{\delta}-1}\right)$$

$$\leq \exp\left(\delta \log x + \frac{1}{\delta} \sum_{p\leq y} \frac{1}{\log p}\right) \leq \exp\left(\delta \log x + \frac{2y}{\delta \log^2 y}\right)$$

by (4.1). Now put $y = \log x \operatorname{loglog} x$ and set $\delta = (\operatorname{loglog} x)^{-\frac{1}{2}}$ and we see that the result follows.

Lemma 2. Let a and k satisfy (1.1) and suppose that $k < y \leq x$ and $z \geq 2$. Then

$$\Phi(x,y,z;k,a) \leq \frac{y}{\phi(k)\log z} + z^2 .$$

This can be proved using the simplest Selberg upper bound sieve method; see [10], p.104.

Lemma 3. Let $f \in M$. Then as $x \to \infty$,

$$\sum_{\substack{n\leq x\\ (n,k)=1}} \frac{f(n)}{n} << \exp\left(\sum_{\substack{p\leq x\\ p\nmid k}} \frac{f(p)}{p}\right) .$$

uniformly in k.

Proof. Since $f \in M$, we have, by (2.3), that

$$\sum_{\substack{n\leq x\\(n,k)=1}} \frac{f(n)}{n} \leq \prod_{\substack{p\leq x\\p\nmid k}} \left(1 + \frac{f(p)}{p} + \sum_{\ell=2}^{\infty} \frac{f(p^{\ell})}{p^{\ell}}\right)$$

$$\leq \exp\left\{\sum_{\substack{p\leq x\\p\nmid k}} \left(\frac{f(p)}{p} + \sum_{\ell=2}^{\infty} \frac{f(p^{\ell})}{p^{\ell}}\right)\right\}$$

$$= \exp\left\{\sum_{\substack{p\leq x\\p\nmid k}} \frac{f(p)}{p} + O(1)\right\} << \exp\left(\sum_{\substack{p\leq x\\p\nmid k}} \frac{f(p)}{p}\right).$$

Lemma 4. Let $f \in M$. Then, as $z \to \infty$

$$\sum_{\substack{n\geq z^{1/2}\\p(n)\leq z^{1/r}\\(n,k)=1}} \frac{f(n)}{n} << \exp\left(\sum_{\substack{p\leq z\\p\nmid k}} \frac{f(p)}{p} - \frac{1}{10}\, r \log r\right)$$

uniformly in k and r, provided that $1 \leq r \leq \dfrac{\log z}{\operatorname{loglog} z}$.

This lemma corresponds to Lemma 3 in [39] except that we generalise it to incorporate the condition $(n,k) = 1$, and replace the implicit constant for the coefficient of $r \log r$ by an absolute constant. Since it is crucial for our application that the constants involved are independent of k, and out class M is larger than Wolke's, we shall give the detailed proof here.

Proof. Again we use Rankin's method. Let $\frac{3}{4} \leq \delta \leq 1$. We have, by (2.3), that

$$\sum_{\substack{n\geq x\\ p(n)\leq y\\ (n,k)=1}} \frac{f(n)}{n} \leq x^{\delta-1} \sum_{\substack{n\geq x\\ p(n)\leq y\\ (n,k)=1}} \frac{f(n)}{n^{\delta}} \leq x^{\delta-1} \sum_{\substack{n\geq 1\\ p(n)\leq y\\ (n,k)=1}} \frac{f(n)}{n^{\delta}}$$

$$= x^{\delta-1} \prod_{\substack{p\leq y\\ p\nmid k}} \left(1 + \frac{f(p)}{p^{\delta}} + \sum_{\ell=2}^{\infty} \frac{f(p^{\ell})}{p^{\ell\delta}}\right)$$

$$\leq \exp\left\{(\delta-1)\log x + \sum_{\substack{p\leq y\\ p\nmid k}} \frac{f(p)}{p^{\delta}} + O(1)\right\} .$$

Now $\dfrac{f(p)}{p^{\delta}} = \dfrac{f(p)}{p} + \dfrac{f(p)}{p}(p^{1-\delta}-1)$, and, by condition (i),

$$\sum_{p\leq y} \frac{f(p)}{p}(p^{1-\delta}-1) \leq \sum_{p\leq y} \frac{A_1}{p} \sum_{n=1}^{\infty} \frac{((1-\delta)\log p)^n}{n!}$$

$$\leq A_1 \sum_{n=1}^{\infty} \frac{(1-\delta)^n \log^{n-1} y}{n!} \sum_{p\leq y} \frac{\log p}{p}$$

$$\leq 2A_1 \sum_{n=1}^{\infty} \frac{((1-\delta)\log y)^n}{n!} \leq 2A_1 \exp\left\{(1-\delta)\log y\right\}$$

$$= 2A_1\, y^{1-\delta} .$$

Therefore we have

$$\sum_{\substack{n\geq x\\ p(n)\leq y\\ (n,k)=1}} \frac{f(n)}{n} \ll \exp\left\{(\delta-1)\log x + \sum_{\substack{p\leq y\\ p\nmid k}} \frac{f(p)}{p} + 2A_1\, y^{1-\delta}\right\}.$$

Now put $x = z^{\frac{1}{2}}$, $y = z^{\frac{1}{r}}$ and set $\delta = 1 - \dfrac{r \log r}{4 \log z}$.

Note that if $1 \leq r \leq \log z/\log\log z$, then $r \log r < \log z$ and so $\frac{3}{4} < \delta \leq 1$. Moreover, we now have

$$(1-\delta)\log x = \frac{r \log r}{4 \log z} \cdot \frac{1}{2} \log z = \frac{1}{8}\, r \log r$$

and

$$y^{1-\delta} = z^{\frac{1-\delta}{r}} = z^{\frac{\log r}{4 \log z}} = r^{\frac{1}{4}}.$$

We now have

$$\sum_{\substack{n\geq z^{1/2}\\ p(n)\leq z^{1/r}\\ (n,k)=1}} \frac{f(n)}{n} \ll \exp\left(\sum_{\substack{p\leq z^{1/r}\\ p\nmid k}} \frac{f(p)}{p} - \frac{1}{8}\, r \log r + 2A_1\, r^{\frac{1}{4}}\right)$$

and the required result follows at once.

3.5 PROOF OF THE MAIN THEOREM.

Let k and y satisfy (2.2) and put

$$z = y^{\frac{\alpha}{10}}. \tag{5.1}$$

For each n satisfying $x - y < n \leq x$, $n \equiv a \pmod k$ we express n in the form

$$n = p_1^{s_1} \cdots p_j^{s_j}\, p_{j+1}^{s_{j+1}} \cdots p_\ell^{s_\ell} = b_n\, d_n, \quad (p_1 < p_2 < \cdots < p_\ell),$$

where b_n is chosen so that

$$b_n \leqslant z < b_n \, p_{j+1}^{s_{j+1}} \quad . \tag{5.2}$$

We devide the set of such integers n into the following classes:

I. $q(d_n) > z^{\frac{1}{2}}$;

II. $q(d_n) \leqslant z^{\frac{1}{2}}, \quad b_n \leqslant z^{\frac{1}{2}}$;

III. $q(d_n) \leqslant \log x \operatorname{loglog} x, \quad b_n > z^{\frac{1}{2}}$;

IV. $\log x \operatorname{loglog} x < q(d_n) \leqslant z^{\frac{1}{2}}, \quad b_n > z^{\frac{1}{2}}$.

First we have

$$\sum_{n\in I} f(n) = \sum_{n\in I} f(b_n)\, f(d_n) \leqslant \sum_{b\leqslant z} f(b) \sum_{\substack{x-y<n\leqslant x \\ n\equiv a \pmod k \\ n\equiv 0 \pmod b \\ q\left(\frac{n}{b}\right)>z^{1/2}}} f\left(\frac{n}{b}\right)$$

$$= \sum_{\substack{b\leqslant z \\ (b,k)=1}} f(b) \sum_{\substack{\frac{x}{b}-\frac{y}{b}<d\leqslant\frac{x}{b} \\ d\equiv a' \pmod k \\ q(d)>z^{1/2}}} f(d) \quad ,$$

where $a' \equiv a\bar{b}, \quad b\bar{b} \equiv 1 \pmod k$. From (5.1) we have

$$q(d) > z^{\frac{1}{2}} = y^{\frac{\alpha}{20}} > x^{\frac{\alpha\beta}{20}} \quad ,$$

so that

$$\Omega(d) \leqslant \frac{\log x}{\log q(d)} < \frac{20}{\alpha\beta} \quad ,$$

and hence, by condition (i),

$$f(d) \leqslant A_1^{\Omega(d)} \leqslant A_1^{\frac{\alpha\beta}{20}} \quad .$$

We have therefore

$$\sum_{n\in I} f(n) \ll \sum_{\substack{b\leq z\\(b,k)=1}} f(b)\ \Phi\left(\frac{x}{b}, \frac{y}{b}, z^{\frac{1}{2}};k,a'\right).$$

Since $k < y^{1-\alpha}$ and $b \leq z < y^{\alpha}$, so that $kb < y$, it follows from Lemma 2 that

$$\Phi\left(\frac{x}{b}, \frac{y}{b}, z^{\frac{1}{2}};k,a'\right) \leq \frac{2y}{\phi(k)b \log z} + z,$$

and therefore

$$\sum_{n\in I} f(n) \ll \left\{\frac{y}{\phi(k)\log z} + z^2\right\} \sum_{\substack{b\leq z\\(b,k)=1}} \frac{f(b)}{b}.$$

From Lemma 3 we arrive at

$$\sum_{n\in I} f(n) \ll \left\{\frac{y}{\phi(k)\log z} + z^2\right\} \exp\left(\sum_{\substack{p\leq z\\p\nmid k}} \frac{f(p)}{p}\right). \tag{5.3}$$

Next, to each $n \in II$, there correspond p and s such that $p^s || n$, $p \leq z^{\frac{1}{2}}$ and by (5.2) $p^s > z^{\frac{1}{2}}$. Let s_p denote the least positive integer s satisfying $p^s > z^{\frac{1}{2}}$ so that $s_p \geq 2$ and hence $p^{-s_p} \leq \min(z^{-\frac{1}{2}}, p^{-2})$; thus

$$\sum_{p\leq z^{1/2}} \frac{1}{p^{s_p}} \leq \sum_{p\leq z^{1/4}} z^{-\frac{1}{2}} + \sum_{p>z^{1/4}} p^{-2} \ll z^{-\frac{1}{4}}.$$

It now follows that

$$\sum_{n\in II} 1 \leqslant \sum_{p\leqslant z^{1/2}} \sum_{\substack{x-y<n\leqslant x \\ n\equiv a(\mathrm{mod}\ k) \\ n\equiv 0(\mathrm{mod}\ p\ \ )}} 1$$

$$= \sum_{\substack{p\leqslant z^{1/2} \\ p\nmid k}} \left\{\frac{y}{kp^{s_p}} + O(1)\right\} << \frac{y}{k} z^{-\frac{1}{4}} + z^{\frac{1}{2}} . \qquad (5.4)$$

Suppose next that $n \in III$. Then there exists b such that

$$b|n, \quad z^{\frac{1}{2}} < b \leqslant z,$$

and

$$p(b) < \log x \ \mathrm{loglog}\ x.$$

Consequently we have

$$\sum_{n\in III} 1 \leqslant \sum_{\substack{z^{1/2}<b\leqslant z \\ p(b)<\log x\ \mathrm{loglog}\ x}} \sum_{\substack{x-y<n\leqslant x \\ n\equiv a(\mathrm{mod}\ k) \\ n\equiv 0(\mathrm{mod}\ b)}} 1$$

$$= \sum_{\substack{z^{1/2}<b\leqslant z \\ p(b)<\log x\ \mathrm{loglog}\ x \\ (b,k)=1}} \left\{\frac{y}{kb} + O(1)\right\}$$

$$\leqslant \frac{y}{k} z^{-\frac{1}{2}} \Psi(z, \log x\ \mathrm{loglog}\ x) + O(z) << \frac{y}{k} z^{-\frac{1}{4}} + z \qquad (5.5)$$

by Lemma 1. Since $k < y^{1-\alpha}$ we have, by (5.1),

$$z < y^{\alpha} z^{-\frac{1}{4}} < \frac{y}{k} z^{-\frac{1}{4}} ,$$

and so, from (5.4) and (5.5) we have

$$\sum_{n\in II} 1 + \sum_{n\in III} 1 \ll \frac{y}{k} z^{-\frac{1}{4}} .$$

For $y > x^{\beta}$ we have, by condition (ii) and (5.1), that

$$f(n) \ll n^{\frac{\alpha\beta}{80}} \leq x^{\frac{\alpha\beta}{80}} < y^{\frac{\alpha}{80}} = z^{\frac{1}{8}}$$

so that we arrive at

$$\sum_{n\in II} f(n) + \sum_{n\in III} f(n) \ll \frac{y}{k} z^{-\frac{1}{8}} . \tag{5.6}$$

Lastly we deal with the class IV. We have

$$\sum_{n\in IV} f(n) = \sum_{n\in IV} f(b_n) f(d_n) \leq \sum_{z^{1/2}<b\leq z} f(b) \sum_{\substack{x-y<n\leq x \\ n\equiv a(\text{mod } k) \\ b_n=b,q(d_n)>p(b) \\ \log x \log\log x<q(d_n)\leq z^{1/2}}} f(d_n) .$$

Let us put

$$r_o = \left[\frac{\log z}{\log(\log x \log\log x)}\right] , \tag{5.7}$$

so that $\log x \log\log x > z^{\frac{1}{r_o+1}}$. Let $2 \leq r \leq r_o$ and consider those

n for which $z^{\frac{1}{r+1}} < q(d_n) \leq z^{\frac{1}{r}}$.

For such n, we have $p(b_n) = p(b) < q(d_n) < z^{\frac{1}{r}}$ and moreover, as before,

$$\Omega(d_n) \leqslant \frac{\log x}{\log q(d_n)} \leqslant \frac{(r+1)\log x}{\log x} < \frac{10(r+1)}{\alpha\beta} < \frac{20r}{\alpha\beta}$$

so that $f(d_n) \leqslant A_1^{\Omega(d_n)} \leqslant A_5^r$, $\quad A_5 = A_1^{\frac{20}{\alpha\beta}}$.

It follows that

$$\sum_{n\in IV} f(n) \leqslant \sum_{2\leqslant r\leqslant r_o} A_5^r \sum_{\substack{z^{1/2}<b\leqslant z \\ p(b)<z^{1/r}}} f(b) \sum_{\substack{x-y<n\leqslant x \\ n\equiv a \pmod{k} \\ n\equiv 0 \pmod{b} \\ z^{1/(r+1)}<q\left(\frac{n}{b}\right)<z^{1/r}}} 1$$

$$\leqslant \sum_{2\leqslant r\leqslant r_o} A_5^r \sum_{\substack{z^{1/2}<b\leqslant z \\ p(b)<z^{1/r} \\ (b,k)=1}} f(b)\, \Phi\left(\frac{x}{b}, \frac{y}{b}, z^{\frac{1}{r+1}}; k,a'\right)$$

where $a' \equiv a\bar{b}$, $b\bar{b} \equiv 1 \pmod{k}$. From Lemma 2 we have

$$\Phi\left(\frac{x}{b}, \frac{y}{b}, z^{\frac{1}{r+1}}; k,a'\right) \leqslant \frac{y(r+1)}{\phi(k)\ b \log z} + z^{\frac{2}{r+1}} ,$$

and therefore

$$\sum_{n\in IV} f(n) \leqslant \left(\frac{y}{\phi(k)\log z} + z^2\right) \sum_{2\leqslant r\leqslant r_o} (r+1)\, A_5^r \sum_{\substack{z^{1/2}<b\leqslant z \\ p(b)<z^{1/r} \\ (b,k)=1}} \frac{f(b)}{b} .$$

From (5.7) we see that we can apply Lemma 4 to the inner sum here giving

$$\sum_{n\in IV} f(n) \ll \left(\frac{y}{\phi(k)\log z} + z^2\right) \exp\left(\sum_{\substack{p\leq z\\ p\nmid k}} \frac{f(p)}{p}\right) \sum_{2\leq r\leq r_0} r\, A_5^r \exp\left(-\frac{1}{10} r \log r\right)$$

$$\ll \left(\frac{y}{\phi(k)\log z} + z^2\right) \exp\left(\sum_{\substack{p\leq z\\ p\nmid k}} \frac{f(p)}{p}\right) . \tag{5.8}$$

For $k < y^{1-\alpha}$ we have, by (5.1),

$$z^2 < \frac{k}{\phi(k)} \frac{z^3}{\log z} < \frac{y^{1-\alpha+3\alpha/10}}{\phi(k)\log z} < \frac{y}{\phi(k)\log z} .$$

From (5.3) and (5.8) we now have

$$\sum_{n\in I} f(n) + \sum_{n\in IV} f(n) \ll \frac{y}{\phi(k)\log z} \exp\left(\sum_{\substack{p\leq z\\ p\nmid k}} \frac{f(p)}{p}\right)$$

and this, together with (5.6), gives the desired result (2.1).

3.6 APPLICATIONS OF THE MAIN THEOREM.

It is easy to see that the expected result (1.3) is an immediate consequence of our main theorem; indeed we have:

Theorem 2. Let α, β, λ be real numbers and let a, k, r be integers. Suppose that

$$0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{2}, \quad r \geq 2, \quad 0 < a < k, \quad (a,k) = 1.$$

We have, as $x \to \infty$,

$$\sum_{\substack{x-y<n\leq x \\ n\equiv a(\mathrm{mod}\ k)}} d_r^\lambda(n) \ll \frac{y}{k}\left(\frac{\phi(k)}{k}\log x\right)^{r^\lambda-1}$$

uniformly in a, k and y provided that

$$k < y^{1-\alpha}, \qquad x^\beta < y \leq x .$$

Proof. We put

$$f(n) = d_r^\lambda(n)$$

so that f(n) is multiplicative. Also, given any fixed r there exists c = c(r) such that

$$d_r(p^\ell) \leq c\,\ell^{r-1}, \qquad p \text{ prime}, \qquad \ell \leq 1 ,$$

so that (1.4) holds, and hence $f \in M$. Next we have

$$f(p) = d_r^\lambda(p) = r^\lambda$$

so that, for $k < y^{1-\alpha}$,

$$\sum_{\substack{p\leq x \\ p\nmid k}} \frac{f(p)}{p} = r^\lambda\left\{\sum_{p\leq x}\frac{1}{p} - \sum_{p|k}\frac{1}{p}\right\}$$

$$= r^\lambda\left\{\log\log x - \sum_{p|k}\frac{1}{p} + O(1)\right\}$$

where the implied constant is absolute. Since

$$\exp\left(-r^\lambda\sum_{p|k}\frac{1}{p}\right) = \prod_{p|k}\exp\left(-\frac{r^\lambda}{p}\right)$$

$$\ll \prod_{\substack{p|k \\ p>r^\lambda}} \left(1 - \frac{r^\lambda}{p}\right) \ll \prod_{p|k} \left(1 - \frac{1}{p}\right)^{r^\lambda}$$

$$= \left\{\frac{\phi(k)}{k}\right\}^{r^\lambda}$$

we see that

$$\exp\left(\sum_{\substack{p \leqslant x \\ p|k}} \frac{f(p)}{p}\right) \ll \left\{\frac{\phi(k)}{k} \log x\right\}^{r^\lambda}$$

where the implied constant depends on r and λ. The required result therefore follows from Theorem 1.

Smith [34] has obtained an asymptotic formula for the sum

$$\sum_{\substack{n \leqslant x \\ n \equiv a (\mathrm{mod}\ k)}} r(n)$$

valid for $k < x^{2/3-\alpha}$. Here we have:

<u>Theorem 3.</u> <u>Let</u> α, β, λ <u>be real numbers and let</u> a, k <u>be integers.</u> <u>Suppose that</u>

$$0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{2}, \quad 0 < a < k, \quad (a,k) = 1.$$

<u>We have</u>, <u>as</u> $x \to \infty$,

$$\sum_{\substack{x-y<n \leqslant x \\ n\equiv a (\mathrm{mod}\ k) \\ r(n)>0}} r^\lambda(n) \ll \frac{y}{\phi(k)} \prod_{\substack{p|k \\ p\equiv 1 (\mathrm{mod}\ 4) \\ p>2^\lambda}} \left(1 - \frac{2^\lambda}{p}\right) (\log x)^{2^{\lambda-1}-1} \tag{6.1}$$

<u>uniformly in</u> a,k <u>and</u> y <u>provided that</u>

$$k < y^{1-\alpha} , \qquad x^{\beta} < y \leq x .$$

Note that if we put $\lambda = 0$, then (6.1) becomes

$$\sum_{\substack{x-y<n\leq x \\ n\equiv a (\text{mod } k) \\ r(n)>0}} 1 << \frac{1}{\phi(k)} \prod_{\substack{p|k \\ p\equiv 1 (\text{mod } 4)}} \left(1 - \frac{1}{p}\right) \cdot \frac{y}{\sqrt{\log x}} .$$

Proof of Theorem 3. Here we let

$$f(n) = \begin{cases} \left(\frac{r(n)}{4}\right)^{\lambda} & \text{if} \quad r(n) > 0 , \\ 0 & \text{if} \quad r(n) = 0 , \end{cases}$$

so that f(n) is multiplicative and it is clear that $f \in M$.

We have

$$f(p) = \begin{cases} 2^{\lambda} & \text{if} \quad p \equiv 1 \ (\text{mod } 4) \\ 0 & \text{if} \quad p \equiv 3 \ (\text{mod } 4) \\ 1 & \text{if} \quad p = 2 . \end{cases}$$

Also, by Merten's theorem,

$$\sum_{\substack{p\leq x \\ p\equiv 1 (\text{mod } 4)}} \frac{1}{p} = \frac{1}{2} \log\log x + O(1), \qquad x \to \infty ,$$

so that we have, for $k < y^{1-\alpha}$,

$$\sum_{\substack{p\leq x \\ p\nmid k}} \frac{f(p)}{p} = 2^{\lambda} \left\{\frac{1}{2} \log\log x - \sum_{\substack{p|k \\ p\equiv 1 (\text{mod } 4)}} \frac{1}{p} + O(1)\right\} .$$

Now

$$\exp\left\{-\sum_{\substack{p|k\\ p\equiv 1 (\mathrm{mod}\ 4)}} \frac{2\lambda}{p}\right\} = \prod_{\substack{p|k\\ p\equiv 1 (\mathrm{mod}\ 4)}} \exp\left(-\frac{2\lambda}{p}\right)$$

$$\ll \prod_{\substack{p|k\\ p\equiv 1 (\mathrm{mod}\ 4)\\ p>2^{\lambda}}} \left(1 - \frac{2\lambda}{p}\right)$$

and the required result follows from Theorem 1.

Let $\delta(n)$ denote the number of square full divisors of n. Knopfmacher [15] has obtained an asymptotic formula for the sum

$$\sum_{\substack{n\leq x\\ n\equiv a (\mathrm{mod}\ k)}} \delta(n)$$

valid for $k < x^{7/36-\alpha}$. As our last application of our main theorem we have:

Theorem 4. Let α, β, λ be real numbers and a, k be integers. Suppose that

$$0 < \alpha < \frac{1}{2}, \qquad 0 < \beta < \frac{1}{2}, \qquad 0 < a < k, \qquad (a,k) = 1.$$

We have, as $x \to \infty$,

$$\sum_{\substack{x-y<n\leq x\\ n\equiv a (\mathrm{mod}\ k)}} \delta^{\lambda}(n) \ll \frac{y}{k}$$

uniformly in a,k and y provided that

$$k < y^{1-\alpha}, \qquad x^{\beta} < y \leq x .$$

Proof. Let $f(n) = \delta^{\lambda}(n)$. Then clearly $f \in M$ and $f(p) = 1$ for all p. As in the proof of Theorem 2, we have

$$\exp\left(\sum_{\substack{p\leq x\\ p\nmid k}} \frac{1}{p}\right) \ll \frac{\phi(k)}{k}\log x\,, \qquad k < y^{1-\alpha}$$

and so the required result follows from Theorem 1.

3.7 A GENERALISATION OF TURAN'S LEMMA.

We shall prove the following generalisation of Turan's lemma.

Theorem 5. *Let $0 < \alpha < \frac{1}{2}$ and a, k, ℓ be integers satisfying*

$$0 < a < k\,, \qquad (a,k) = 1, \qquad \ell > 0.$$

Let $f(n)$ denote either $w(n)$ or $\Omega(n)$. Then, as $x \to \infty$,

$$\sum_{\substack{n\leq x\\ n\equiv a \pmod k}} f^{\ell}(n) = \frac{x}{k}(\log\log x)^{\ell} + O\left(\frac{x}{k}(\log\log x)^{\ell-1}\log\log\log x\right), \tag{7.1}$$

uniformly in a and k provided that $k < x^{1-\alpha}$.

Remarks.

(i) The implied constant depends on ℓ and α.

(ii) If k varies boundedly the error term in (7.1) is only $O(x(\log\log x)^{\ell-1})$. This is clear from the proof.

Proof. Consider first the case $\ell = 1$. For $n \leq x$ we have

$$w(n) = \sum_{p|n} 1 = \sum_{\substack{p|n \\ p \leq x^{\alpha}}} 1 + O(1)$$

since

$$\sum_{\substack{p|n \\ p > x^{\alpha}}} 1 < \frac{1}{\alpha} .$$

It follows that

$$\sum_{\substack{n \leq x \\ n \equiv a (\bmod k)}} w(n) = \sum_{\substack{n \leq x \\ n \equiv a (\bmod k)}} \left\{ \sum_{\substack{p|n \\ p \leq x^{\alpha}}} 1 + O(1) \right\}$$

$$= \sum_{p \leq x^{\alpha}} \sum_{\substack{n \leq x \\ n \equiv a (\bmod k) \\ n \equiv 0 (\bmod p)}} 1 + O\left(\frac{x}{k}\right)$$

$$= \sum_{\substack{p \leq x\alpha \\ p \nmid k}} \sum_{\substack{m \leq x/p \\ m \equiv a' (\bmod k)}} 1 + O\left(\frac{x}{k}\right)$$

where $a' \equiv ab$, $bp \equiv 1 \pmod k$. Since $x^{\alpha} < x/k$ we now have

$$\sum_{\substack{n \leq x \\ n \equiv a (\bmod k)}} w(n) = \sum_{\substack{p \leq x^{\alpha} \\ p \nmid k}} \left\{ \frac{x}{pk} + O(1) \right\} + O\left(\frac{x}{k}\right)$$

$$= \frac{x}{k} \sum_{\substack{p \leq x^{\alpha} \\ p \nmid k}} \frac{1}{p} + O\left(\frac{x}{k}\right) . \qquad (7.2)$$

By writing $n = n_1 n_2$ where $p(n_2) \leq x^{\alpha} < q(n_1)$ we see that

$$\Omega(n) = \Omega(n_2) + O(1)$$

so that

$$\sum_{\substack{n\leq x\\ n\equiv a(\mathrm{mod}\ k)}} \Omega(n) = \sum_{\substack{n\leq x\\ n\equiv a(\mathrm{mod}\ k)}} \Omega(n_2) + O\left(\frac{x}{k}\right)$$

$$= \sum_{\substack{n\leq x\\ n\equiv a(\mathrm{mod}\ k)}} \left\{ \sum_{m\leq \frac{\log x}{\log 2}} \sum_{\substack{p\leq x^{\alpha}\\ p^m|n}} 1 \right\} + O\left(\frac{x}{k}\right)$$

$$= \sum_{m\leq \frac{\log x}{\log 2}} \sum_{p\leq x^{\alpha}} \sum_{\substack{n\leq x\\ n\equiv a(\mathrm{mod}\ k)\\ n\equiv 0(\mathrm{mod}\ p^m)}} 1 + O\left(\frac{x}{k}\right)$$

$$= \sum_{m\leq \frac{\log x}{\log 2}} \sum_{\substack{p\leq x^{\alpha}\\ p\nmid k}} \left\{ \frac{x}{p^m k} + O(1) \right\} + O\left(\frac{x}{k}\right) .$$

Now

$$\sum_{2\leq m\leq \frac{\log x}{\log 2}} \sum_{\substack{p\leq x^{\alpha}\\ p\nmid k}} \left\{ \frac{x}{p^m k} + O(1) \right\} << \frac{x}{k} \sum_{p} \sum_{m\geq 2} \frac{1}{p^m} + \pi(x^{\alpha})\log x$$

$$<< \frac{x}{k} + x^{\alpha} << \frac{x}{k}$$

so that

$$\sum_{\substack{n\leq x\\ n\equiv a(\mathrm{mod}\ k)}} \Omega(n) = \frac{x}{k} \sum_{\substack{p\leq x^{\alpha}\\ p\nmid k}} \frac{1}{p} + O\left(\frac{x}{k}\right)$$

and, together with (7.2) we have

$$\sum_{\substack{n \le x \\ n \equiv a \pmod{k}}} f(n) = \frac{x}{k} \sum_{\substack{p \le x^{\alpha} \\ p \nmid k}} \frac{1}{p} + O\left(\frac{x}{k}\right) . \tag{7.3}$$

Now

$$\sum_{\substack{p \le x^{\alpha} \\ p \nmid k}} \frac{1}{p} = \sum_{\substack{p \le x \\ p \nmid k}} \frac{1}{p} + O(1)$$

$$= \sum_{p \le x} \frac{1}{p} + O\left(\sum_{p | k} \frac{1}{p}\right) + O(1) ,$$

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1) ,$$

and

$$\sum_{p | k} \frac{1}{p} \le \sum_{p \le \log x} \frac{1}{p} + \sum_{\substack{p | k \\ p > \log x}} \frac{1}{p}$$

$$<< \log\log\log x + \frac{w(k)}{\log x}$$

$$<< \log\log\log x ,$$

since $w(k) << \log k < \log x$. The required result (7.1) therefore follows from (7.3).

We next proceed to prove the general case by induction on ℓ. We use as induction hypothesis that (7.1) holds uniformly in a and k provided that $k < x^{1-\alpha/2}$. Suppose now that $k < x^{1-\alpha}$ and we consider

$$\sum_{\substack{n \le x \\ n \equiv a \pmod{k}}} (w(n))^{\ell+1} = \sum_{\substack{n \le x \\ n \equiv a \pmod{k}}} w^{\ell}(n) \left\{ \sum_{\substack{p | n \\ p \le x^{\alpha/2}}} 1 + O(1) \right\}$$

$$= \sum_{p \leq x^{\alpha/2}} \sum_{\substack{n \leq x \\ n \equiv a \pmod{k} \\ n \equiv 0 \pmod{p}}} w^{\ell}(n) + O\left(\frac{x}{k} (\text{loglog } x)^{\ell}\right)$$

$$= \sum_{\substack{p \leq x^{\alpha/2} \\ p \nmid k}} \sum_{\substack{m \leq x/p \\ m \equiv a' \pmod{k}}} w^{\ell}(mp) + O\left(\frac{x}{k} (\text{loglog } x)^{\ell}\right), \tag{7.4}$$

where $a' \equiv ab$, $bp \equiv 1 \pmod{k}$. Now

$$w^{\ell}(mp) = w^{\ell}(m) + O\left((w(m))^{\ell-1}\right), \tag{7.5}$$

and if $p \leq x^{\alpha/2}$, $k < x^{1-\alpha}$, then $k < (x/p)^{1-\alpha/2}$ so that, by the induction hypothesis, we have

$$\sum_{\substack{m \leq x/p \\ m \equiv a' \pmod{k}}} w^{\ell}(m) = \frac{x}{pk} \left(\text{loglog } \frac{x}{p}\right)^{\ell} + O\left(\frac{x}{pk} (\text{loglog } x)^{\ell-1} \text{ logloglog } x\right)$$

$$= \frac{x}{pk} \left(\text{loglog } x\right)^{\ell} + O\left(\frac{x}{pk} (\text{loglog } x)^{\ell-1} \text{ logloglog } x\right). \tag{7.6}$$

Also

$$\sum_{\substack{m \leq x/p \\ m \equiv a' \pmod{k}}} w^{\ell-1}(m) \ll \frac{x}{pk} (\text{loglog } x)^{\ell-1} \tag{7.7}$$

uniformly in $k < x^{1-\alpha}$, $p \leq x^{\alpha/2}$. As before we have

$$\sum_{\substack{p \leq x^{\alpha} \\ p \nmid k}} \frac{1}{p} = \text{loglog } x + O(\text{logloglog } x)$$

uniformly in $k < x$, so that from (7.4), (7.5), (7.6) and (7.7) we have

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod{k}}} (w(n))^{\ell+1} = \frac{x}{k} (\text{loglog } x)^{\ell+1} + O\left(\frac{x}{k} (\text{loglog } x)^{\ell} \text{ logloglog } x\right)$$

uniformly in $k < x^{1-\alpha}$.

The same result is obtained for $\Omega(n)$ by applying the method used for $\ell = 1$. The inductive step is complete and so the theorem is proved.

Chapter Four

The Distribution of Power Full Integers

4.1 INTRODUCTION.

Let k be an integer greater than 1. We call a positive integer n a k-full integer if p^k divides n whenever p is a prime divisor of n. It is clear that each k-full integer can be written in the form

$$a_1^k \, a_2^{k+1} \cdots a_k^{2k-1}$$

where each a_i is a positive integer; moreover, this representation is unique if we stipulate that $a_2 \ldots a_k$ is square free.

For $x \geqslant 1$ we denote by $Q_k(x)$ the number of k-full integers not exceeding x so that

$$Q_k(x) = \sum_{a_1^k \ldots a_k^{2k-1} \leqslant x} \mu^2(a_2 \ldots a_k) \tag{1.1}$$

where $\mu(n)$ is the Möbius function. We shall see that $Q_k(x)$ is related to the corresponding unweighted sum

$$S_k(x) = \sum_{a_1^k \ldots a_k^{2k-1} \leqslant x} 1 \tag{1.2}$$

which, following standard procedures, satisfies

$$S_k(x) = \sum_{k \leqslant r < 2k} A^*_{kr} \, x^{\frac{1}{r}} + \Delta^*_k(x) \tag{1.3}$$

where

$$A^{*}_{kr} = \prod_{\substack{k\leqslant n<2k \\ n\neq r}} \zeta\left(\frac{n}{r}\right), \qquad k \leqslant r < 2k , \tag{1.4}$$

and $\Delta^{*}_{k}(x)$ is an error term. Let us define ρ^{*}_{k} to be the infimum of all ρ satisfying

$$\Delta^{*}_{k}(x) << x^{\rho} , \qquad x \to \infty .$$

From the application of a special case of the colossal lattice point theorem due to Landau [17], [18], we can show that

$$\frac{k-1}{k(3k-1)} \leqslant \rho^{*}_{k} \leqslant \frac{1}{k+2} , \qquad k \geqslant 2 . \tag{1.5}$$

If we write

$$A_{kr} = A^{*}_{kr} J_{k}\left(\frac{1}{r}\right), \qquad k \leqslant r < 2k \tag{1.6}$$

where $J_k(s)$ is a function defined later in section 4.2, then we can write

$$Q_{k}(x) = \sum_{k\leqslant r<2k} A_{kr} x^{\frac{1}{r}} + \Delta_{k}(x) \tag{1.7}$$

where $\Delta_k(x)$ is an error term. We define ρ_k to be the infimum of all ρ satisfying

$$\Delta_{k}(x) << x^{\rho} , \qquad x \to \infty .$$

In 1934 Erdös and Szekeres [7] proved in an elementary way that

$$\rho_{k} \leqslant \frac{1}{k+1} , \qquad k \geqslant 2 ,$$

and in 1958 Bateman and Grosswald [2] improved this to

$$\rho_k \leqslant \frac{1}{k+2}, \qquad k \geqslant 2.$$

They proved also that $\rho_2 \leqslant \frac{1}{6}$ and

$$\rho_3 \leqslant \frac{7}{46} = 0.157 \ldots \tag{1.8}$$

and, by relating $Q_k(x)$ to the sum

$$\sum_{a_1^k \ldots a_{r+1}^{k+r} \leqslant x} 1, \qquad r = \left[\sqrt{2k}\right]$$

and then appealing to Landau's theorem, they proved that

$$\rho_k \leqslant \max\left(\frac{r}{k(r+2)}, \frac{1}{k+r+1}\right), \qquad r = \left[\sqrt{2k}\right], \quad k \geqslant 4 . \tag{1.9}$$

Here we shall relate $Q_k(x)$ to $S_k(x)$ and prove:

Theorem 1. We have

$$\rho_k \leqslant \max\left(\rho_k^*, \frac{1}{2k+2}\right), \qquad k \geqslant 2 . \tag{1.10}$$

Moreover, if ρ and λ are constants satisfying

$$\rho \geqslant \frac{1}{2k+2}, \qquad \lambda > 0$$

and

$$\Delta_k^*(x) \ll x^\rho \log^\lambda x, \qquad x \to \infty,$$

then

$$\Delta_k(x) \ll x^\rho \log^{\lambda'} x, \qquad x \to \infty$$

where

$$\lambda' = \begin{cases} \lambda & \text{if} \quad \rho > \frac{1}{2k+2} , \\ \lambda+1 & \text{if} \quad \rho = \frac{1}{2k+2} . \end{cases}$$

From (1.10) we see that the inequality

$$\rho_k^* < \frac{1}{2k+2} \tag{1.11}$$

is of crucial relevance to our upper bound for ρ_k. As a particular case of a general result, Richert [27] has proved that

$$\rho_2^* \leq \frac{2}{15} \tag{1.12}$$

which establishes (1.11) when $k = 2$, the only case settled up to the present. Indeed, from this Bateman and Grosswald proved that

$$\Delta_2(x) << x^{\frac{1}{6}} \exp(-c\,w(x)) , \qquad x \to \infty$$

where

$$w(x) = (\log x)^{\frac{3}{5}} \quad (\log\log x)^{-\frac{1}{5}}$$

and c is a positive constant. They also pointed out that $\rho_2 < \frac{1}{6}$ if and only if the supremum of the real parts of the zeros of the Riemann zeta function is less than 1. It will be clear from the proof of Theorem 1 that if (1.11) holds for any particular k, then we can apply the method of Bateman and Grosswald to prove that

$$\Delta_k(x) << x^{\frac{1}{2k+2}} \exp(-c\,w(x)) , \quad x \to \infty .$$

In section 4.4 we shall apply Theorem 1 together with some results of Schmidt [29] and Srinivasan [35] to prove:

<u>Theorem 2.</u> <u>We have, as $x \to \infty$</u>,

$$\Delta_3(x) \ll x^{\frac{263}{2052}} \log^2 x$$

<u>so that</u>

$$\rho_3 \leqslant \frac{263}{2052} = 0.128 \ldots .$$

Our upper bound for ρ_3 is an improvement on (1.8), and indeed is quite close to $\frac{1}{8}$, the critical value on the right hand side of (1.11) when $k = 3$.

In section 4.5 we shall apply Theorem 1 to give the following improvement on (1.9).

<u>Theorem 3.</u> <u>For $4 \leqslant k \leqslant 12$ we have</u>

$$\Delta_k(x) \ll x^{\frac{1}{2k}} \log^{\frac{3}{2}} x , \qquad x \to \infty ,$$

<u>so that</u>

$$\rho_k \leqslant \frac{1}{2k} , \qquad 4 \leqslant k \leqslant 12 .$$

<u>We also have</u>

$$\rho_k \leqslant \frac{1}{k+u} , \qquad k \geqslant 13 ,$$

<u>where</u>

$$u = \sqrt{6k + \frac{25}{4}} + \frac{5}{2} .$$

Furthermore we have

$$\rho_k \leq \frac{1}{2k}, \qquad k \geq 13$$

on the assumption of the Lindelöf hypothesis.

4.2 THE DIRICHLET SERIES ASSOCIATED WITH $Q_k(x)$.

Let k be fixed and let $\alpha_k(n)$ be the characteristic function of the set of k-full integers. It is clear that $\alpha_k(n)$ is multiplicative and that, for each prime p,

$$\alpha_k(p^a) = \begin{cases} 1 & a = 0, k, k+1, \ldots . \\ 0 & a = 1, 2, \ldots, k-1 . \end{cases}$$

For $s = \sigma + it$, $\sigma > 1/k$ we have that

$$\sum_{n=1}^{\infty} \frac{\alpha_k(n)}{n^s} = \prod_p \left(1 + \sum_{a=k}^{\infty} p^{-as}\right) = \prod_p \left(1 + \frac{p^{-ks}}{1-p^{-s}}\right) . \tag{2.1}$$

Lemma 2.1. Let $k \geq 2$ and $K = \frac{1}{2}(3k^2 + k - 2)$. Then there are constants a_{kr} $(2k + 2 < r \leq K)$ such that

$$\left(1 + \frac{v^k}{1-v}\right)(1-v^k)(1-v^{k+1})\ldots(1-v^{2k-1}) = 1 - v^{2k+2} + \sum_{r=2k+3}^{K} a_{kr} v^r \tag{2.2}$$

holds for all $v \neq 1$.

Proof. The product of the first two factors on the left hand side of (2.2) is

$$(1-v+v^k)(1+v+v^2+\ldots+v^{k+1}) = 1 + v^{k+1} + v^{k+2} + \ldots + v^{2k-1} .$$

It follows that the left hand side of (2.2) is a polynomial in v of degree

$$(2k-1) + (k+1) + (k+2) + \ldots + (2k-1) = K.$$

Moreover, when the product is multiplied out, the terms v^r $(1 \leq r < 2k+2)$ cancel, leaving $-v^{2k+2}$ as the first non-zero term. The lemma is proved.

We now define, for $\sigma > 1/k$,

$$H_k(s) = \prod_{k \leq n < 2k} \zeta(ns) \tag{2.3}$$

and, for $\sigma > 1/(2k+2)$,

$$J_k(s) = \prod_p \left(1 - p^{-(2k+2)s} + \sum_{r=2k+3}^{K} a_{kr}\, p^{-rs}\right) \tag{2.4}$$

so that from (2.1) and (2.2) we have that

$$\sum_{n=1}^{\infty} \frac{\alpha_k(n)}{n^s} = J_k(s)\, H_k(s)\ . \tag{2.5}$$

We note that if $k = 2$, then $K = 6$ and $2k+3 = 7$ so that the sum in the Euler product for $J_2(s)$ is empty, giving $J_2(s) = 1/\zeta(6s)$. We remark that, for $k \geq 3$, the line $\sigma = 0$ is a natural boundary for $J_k(s)$. To see this we need a lemma of Estermann [8] which states that, for small x,

$$1 - x + x^k = \prod_{n=1}^{\infty} (1 - x^n)^{\ell_k(n)}\ . \tag{2.6}$$

Here $\ell_k(n)$ is an integer given by

$$\ell_k(n) = \frac{1}{n} \sum_{ab=n} \mu(a) \sum_{r=1}^{k} \lambda_r^b$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of the equation

$$\lambda^k - \lambda^{k-1} + 1 = 0.$$

From (2.1) and (2.2) we have that

$$J_k(s) = \frac{\zeta(s)}{H_k(s)} \prod_p \left(1 - p^{-s} + p^{-ks}\right)$$

so that from (2.6) we see that $J_k(s)$ can be written as an infinite product of the Riemann zeta functions. For example, we have

(1,0,-1,-1,-1,0,0,1,1,1,0,0,-1,-1,0,0,1,1,1,0,-1,-2,-2,-1,1,3,....)

for the sequence $(\ell_3(n))$ so that

$$J_3(s) = \frac{\zeta(13s)\zeta(14s)\zeta(21s)\zeta^2(22s)\zeta^2(23s)\zeta(24s)\ \ldots}{\zeta(8s)\zeta(9s)\zeta(10s)\zeta(17s)\zeta(18s)\zeta(19s)\zeta(25s)\zeta^3(26s)\ldots} .$$

If we assume the Riemann hypothesis we can deduce easily that the zeros of $J_k(s)$ $(k \geqslant 3)$ are dense in the line $\sigma = 0$. If we follow the proof of the main theorem in Estermann's paper we can give an unconditional proof using only simple zero density estimates for the Riemann zeta function. We shall not require this result in our proofs of the theorems and shall therefore leave it to section 4.6.

We now define $\beta_k(n)$ and $\tau_k(n)$ by

$$J_k(s) = \sum_{n=1}^{\infty} \frac{\beta_k(n)}{n^s} ,$$

and

$$H_k(s) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s}$$

so that, from (2.5), we have

$$\alpha_k = \beta_k * \tau_k . \tag{2.7}$$

We also note that, by (2.3),

$$\tau_k(n) = \sum_{a_1^k \cdots a_k^{2k-1} = n} 1$$

so that

$$S_k(x) = \sum_{n \leq x} \tau_k(n) .$$

Lemma 2.2. *We have, as* $x \to \infty$,

$$\sum_{n \leq x} |\beta_k(n)| << x^{\frac{1}{2k+2}}, \qquad k = 3,4,\ldots .$$

Proof. We see from (2.4) that there are constants a'_{kr} $(2k+2 < r \leq K + 2k+3)$ such that

$$J_k(s) = \frac{1}{\zeta((2k+2)s)} \prod_p \left\{ \frac{1 + \sum_{r=2k+3}^{K+2k+3} a'_{kr}\, p^{-rs}}{1 - p^{-(4k+4)s}} \right\} .$$

It follows that we can write

$$\sum_{n=1}^{\infty} \frac{\beta_k(n)}{n^s} = \sum_{n=1}^{\infty} \frac{h_1(n)}{n^s} \times \sum_{n=1}^{\infty} \frac{h_2(n)}{n^s}$$

where

$$h_1(n) = \begin{cases} \mu(m) & \text{if} \quad n = m^{2k+2} , \\ 0 & \text{otherwise} \end{cases} ,$$

and the series $\Sigma\, h_2(n)/n^s$ converges absolutely in $\sigma > 1/(2k+3)$.

Now

$$\beta_k = h_1 * h_2$$

and, as $y \to \infty$

$$\sum_{n \le y} |h_1(n)| = \sum_{m \le y^{\frac{1}{2k+2}}} \mu^2(m) \ll y^{\frac{1}{2k+2}} .$$

It follows that, as $x \to \infty$,

$$\sum_{n \le x} |\beta_k(n)| = \sum_{ab \le x} |h_1(a)h_2(b)| = \sum_{b \le x} |h_2(b)| \sum_{a \le x/b} |h_1(a)|$$

$$\ll \sum_{b \le x} |h_2(b)| \left(\frac{x}{b}\right)^{\frac{1}{2k+2}} \ll x^{\frac{1}{2k+2}} ,$$

since

$$\sum_{b=1}^{\infty} \frac{|h_2(b)|}{b^{1/(2k+2)}} < \infty .$$

4.3 PROOF OF THEOREM 1.

We have, by (2.7) and (2.8), that

$$Q_k(x) = \sum_{n \le x} \alpha_k(n) = \sum_{mn \le x} \beta_k(m)\, \tau_k(n)$$

$$= \sum_{m \le x} \beta_k(m)\, S_k\left(\frac{x}{m}\right) .$$

From (1.3) we now have that

$$Q_k(x) = \sum_{k\leq r<2k} A_{kr}^* x^{\frac{1}{r}} \sum_{m\leq x} \frac{\beta_k(m)}{m^{1/r}} + \sum_{m\leq x} \beta_k(m)\, \Delta_k^*\left(\frac{x}{m}\right) . \tag{3.1}$$

From Lemma 2.2 we have, by partial summations, that

$$\sum_{m>x} \frac{|\beta_k(m)|}{m^{1/r}} << x^{\frac{1}{2k+2} - \frac{1}{r}} , \qquad (k \leq r < 2k) ,$$

so that

$$\sum_{k\leq r<2k} A_{kr}^* x^{\frac{1}{r}} \sum_{m\leq x} \frac{\beta_k(m)}{m^{1/r}} = \sum_{k\leq r<2k} A_{kr}^* x^{\frac{1}{r}} \left\{ J_k\left(\frac{1}{r}\right) + O\left(x^{\frac{1}{2k+2} - \frac{1}{r}}\right)\right\}$$

$$= \sum_{k\leq r<2k} A_{kr} x^{\frac{1}{r}} + O\left(x^{\frac{1}{2k+2}}\right)$$

by (1.6). It now follows from (1.7) and (3.1) that

$$\Delta_k(x) = \sum_{m\leq x} \beta_k(m)\, \Delta_k^*\left(\frac{x}{m}\right) + O\left(x^{\frac{1}{2k+2}}\right). \tag{3.2}$$

Suppose first that $\rho_k^* < 1/(2k+2)$. We can then choose ρ so that $\rho_k^* < \rho < 1/(2k+2)$. From Lemma 2.2 and partial summations, we have

$$\sum_{m\leq x} \frac{|\beta_k(m)|}{m^{\rho}} << x^{\frac{1}{2k+2} - \rho}$$

so that

$$\sum_{m\leq x} \beta_k(m)\,\Delta_k^*\left(\frac{x}{m}\right) << \sum_{m\leq x} |\beta_k(m)| \left(\frac{x}{m}\right)^{\rho} << x^{\frac{1}{2k+2}}$$

and hence, from (3.2)

$$\Delta_k(x) << x^{\frac{1}{2k+2}} \quad .$$

This proves that $\rho_k \leq 1/(2k+2)$ provided that $\rho_k^* < 1/(2k+2)$.

Suppose next that $\rho_k^* > 1/(2k+2)$ and let ρ and λ be such that $\rho > 1/(2k+2)$, $\lambda > 0$ and

$$\Delta_k^*(x) << x^{\rho} \log^{\lambda} x \, , \qquad x \to \infty \quad .$$

Then

$$\sum_{m\leq x} \beta_k(m)\,\Delta_k^*\left(\frac{x}{m}\right) << x^{\rho} \log^{\lambda} x \sum_{m\leq x} \frac{|\beta_k(m)|}{m^{\rho}} << x^{\rho} \log^{\lambda} x \, ,$$

since the Dirichlet series for $J_k(s)$ converges absolutely in $\sigma > 1/(2k+2)$. From (3.2) we now have

$$\Delta_k(x) << x^{\rho} \log^{\lambda} x \, , \qquad x \to \infty \, ,$$

and in particular we have

$$\rho_k \leq \rho_k^* \qquad \text{if} \qquad \rho_k^* > \frac{1}{2k+2} \quad .$$

It is also clear from the above that if

$$\Delta_k^*(x) << x^{\frac{1}{2k+2}} \log^{\lambda} x \, ,$$

then

$$\Delta_k(x) \ll x^{\frac{1}{2k+2}} \log^{\lambda+1} x.$$

The theorem is proved.

Cohen and Davis [4] pointed out that the uniqueness theorem for Dirichlet series is used in the Batemann-Grosswald proof of $\rho_k \leq 1/(k+2)$, and in [4] they gave another elementary proof by means of some special divisor and totient functions to avoid the use of the uniqueness theorem. Here we give a sketch of a direct and elementary proof of Theorem 1 itself without the use of special functions or the uniqueness theorem. We confine ourselves to the case k = 3. We have

$$Q_3(x) = \sum_{a^3b^4c^5 \leq x} \mu^2(bc) = \sum_{a^3b^4c^5 \leq x} \mu^2(b)\mu^2(c) \sum_{\substack{\ell|b \\ \ell|c}} \mu(\ell)$$

$$= \sum_{\substack{a^3m^4n^5\ell^9 \leq x \\ (mn,\ell)=1}} \mu^2(m)\mu^2(n)\mu(\ell)$$

$$= \sum_{\substack{a^3d^4e^5b^8\ell^9c^{10} \leq x \\ (bcde,\ell)=1}} \mu(b)\mu(c)\mu(\ell)$$

$$= \sum_{\substack{b^8\ell^9c^{10} \leq x \\ (bc,\ell)=1}} \mu(b)\mu(c)\mu(\ell)\, S_3'\left(\frac{x}{b^8\ell^9c^{10}}, \ell\right)$$

say, where

$$S_3'(x,\ell) = \sum_{\substack{a^3b^4c^5 \leq x \\ (bc,\ell)=1}} 1 = \sum_{a^3b^4c^5 \leq x} \sum_{\substack{m|bc \\ m|\ell}} \mu(m)$$

$$= \sum_{m|\ell} \mu(m)\, S_3''(x,m)$$

say, where

$$S_3''(x,m) = \sum_{h|m} \sum_{\substack{a^3b^4c^5 \leqslant x \\ bc\equiv 0 (\mathrm{mod}\ m) \\ (b,m)=h}} 1$$

$$= \sum_{m_1 h = m} S_3'''\left(\frac{x}{h^4 m_1^5}, m_1\right)$$

say, where

$$S_3'''(x,m_1) = \sum_{a^3b^4c^5 \leqslant x} \sum_{\substack{\ell | b \\ \ell | m_1}} \mu(\ell)$$

$$= \sum_{\ell | m_1} \mu(\ell)\, S_3\left(\frac{x}{\ell^4}\right)$$

where $S_3(x)$ is defined by (1.2). Substituting (1.3) into here and working back to $Q_3(x)$ we find that

$$Q_3(x) = \sum_{r=3}^{5} A_{3r}^{*} x^{\frac{1}{r}} Q_3(x, \tfrac{1}{r}) + \Delta_3^{+}(x) \tag{3.3}$$

where

$$Q_3(x, \tfrac{1}{r}) = \sum_{\substack{b^8\ell^9c^{10} \leqslant x \\ (bc,\ell)=1}} \frac{\mu(b)\mu(c)\mu(\ell)}{(b^8\ell^9c^{10})^{1/r}} \sum_{m|\ell} \mu(m) g(m, \tfrac{1}{r}),$$

$$g(m, \tfrac{1}{r}) = \sum_{m_1 h = n} \sum_{\ell | m_1} \frac{\mu(\ell)}{(h^4 m_1^5 \ell^4)^{1/r}},$$

$$\Delta_3^{+}(x) = \sum_{\substack{b^8\ell^9c^{10} \leqslant x \\ (bc,\ell)=1}} \mu(b)\mu(c)\mu(\ell)\Delta_3'\left(\frac{x}{b^8\ell^9c^{10}}, \ell\right), \tag{3.4}$$

and

$$\Delta_3'(x,\ell) = \sum_{m|\ell} \sum_{n|m} \sum_{v|n} \mu(m)\mu(v)\Delta_3^*\left(\frac{x}{v^4}\right) . \tag{3.5}$$

If we write

$$f(\ell, \tfrac{1}{r}) = \frac{\mu(\ell)}{\ell^{9/r}} \sum_{m|\ell} \mu(m) g(m, \tfrac{1}{r}) ,$$

and

$$F(\ell, \tfrac{1}{r}) = f(\ell, \tfrac{1}{r}) \prod_{p|\ell} (1-p^{-8/r})^{-1}(1-p^{-10/r})^{-1} ,$$

then we will arrive at

$$Q_3(x, \tfrac{1}{r}) = \frac{1}{\zeta\left(\frac{8}{r}\right)\zeta\left(\frac{10}{r}\right)} \sum_{\ell=1}^{\infty} F(\ell, \tfrac{1}{r}) + O\left(x^{\frac{1}{8} - \frac{1}{r}}\right) .$$

It can be verified that

$$F(p^a, \tfrac{1}{r}) = 0 , \qquad a = 2,3,\ldots$$

and

$$F(p, \tfrac{1}{r}) = \frac{-p^{-9/r} + p^{-13/r} + p^{-14/r} - p^{-18/r}}{(1-p^{-8/r})(1-p^{-10/r})}$$

so that

$$\sum_{\ell=1}^{\infty} F(\ell, \tfrac{1}{r}) = \prod_p \left(1 + F(p, \tfrac{1}{r})\right) = \zeta\left(\frac{8}{r}\right)\zeta\left(\frac{10}{r}\right) J_3\left(\frac{1}{r}\right) .$$

From (1.6), (1.7) and (3.3) we now have that

$$\Delta_3(x) = \Delta_3^+(x) + O(x^{\frac{1}{8}}),$$

and the required result can be derived from (3.4) and (3.5).

4.4 PROOF OF THEOREM 2.

In [7] Erdös and Szekeres also gave the asymptotic formula associated with the number of Abelian groups of a given order mentioned in Chapter Two, section 2.4. This asymptotic formula was rediscovered by Kendall and Rankin [14] who gave a superior estimate for the error term. This then led to a succession of improvements by Richert [27], Schwarz [30], Schmidt [29], and, more recently, by Srinivasan [35].

We define

$$\psi(x) = x - [x] - \frac{1}{2}, \tag{4.1}$$

and, for positive constants α, β, γ we write, following Richert [27],

$$R(x:\alpha,\beta,\gamma) = \sum_{n\leq x^{\alpha}} \psi\left(\frac{x^{\beta}}{n^{\gamma}}\right) \tag{4.2}$$

and, following Schmidt [29],

$$S_{\alpha,\beta,\gamma}(x) = \sum_{\substack{m^{\alpha+\beta}n^{\gamma}\leq x \\ m>n}} \psi\left(\left(\frac{x}{m^{\beta}n^{\gamma}}\right)^{1/\alpha}\right). \tag{4.3}$$

Richert [27] showed that, for positive constants u, v the error term associated with the sum

$$\sum_{a^u b^v \leq x} 1$$

can be expressed in terms of $R(x;\alpha,\beta,\gamma)$ with α,β,γ depending on u and v. In particular the error term $\Delta_2^*(x)$ associated with our sum $S_2(x)$ is given by

$$\Delta_2^* = - R\left(x; \frac{1}{5}, \frac{1}{2}, \frac{3}{2}\right) - R\left(x; \frac{1}{5}, \frac{1}{3}, \frac{2}{3}\right) + O(1), \tag{4.4}$$

a formula that we shall use later in section 5.2. In order to improve on Richert's result Schmidt [29] had to consider the error term $\Delta_3^{(1)}(x)$ associated with the sum

$$\sum_{ab^2c^3 \leq x} 1$$

and he proved that

$$\Delta_3^{(1)}(x) = - \sum_{(\alpha,\beta,\gamma)} S_{\alpha,\beta,\gamma}(x) + O(x^{\frac{1}{6}}) \tag{4.5}$$

where the summation is over the six permutations of (1,2,3).

For our present problem of the error term $\Delta_3^*(a)$ associated with the sum $S_3(x)$ we have

$$\Delta_3^*(x) = - \sum_{(\alpha,\beta,\gamma)} S_{\alpha,\beta,\gamma}(x) + O(x^{\frac{1}{12}}) \tag{4.6}$$

where the summation is now over the six permutations of (3,4,5). We shall omit the proof of (4.6) since it is, of course, the same as that of (4.5).

Next, in order to estimate $S_{\alpha,\beta,\gamma}(x)$ we shall apply the following result due to Srinivasan [35].

<u>Lemma 4.1.</u> (Srinivasan [35], Main Theorem).

<u>Let</u> ρ, $\sigma > 0$ <u>and</u> (λ_o, λ_1) <u>be any two dimensional exponent pair</u>. <u>Let</u> z, M, N, F <u>satisfy</u>

$$F = z M^{-\rho} N^{-\sigma}, \qquad 1 \ll M \ll F, \qquad 1 \ll N \ll F.$$

Then, for any region D in the rectangle $M < m \leq 2M$, $N < n \leq 2N$, we have

$$\sum_{(m,n)\in D} \psi\left(\frac{z}{m^{\rho} n^{\sigma}}\right) << \left\{F^{\frac{1}{2}+\lambda_o-\lambda_1} M^{\frac{1}{2}+2\lambda_o} N^{\frac{3}{2}-2\lambda_1}\right\}^{\frac{1}{\frac{3}{2}+\lambda_o-\lambda_1}} + F^{\frac{1}{4}} M^{\frac{1}{4}} N + F^{-\frac{1}{2}} M N. \tag{4.7}$$

The definition of a two dimensional exponent pair is given in [35] where it is also shown that

$$\left(\frac{23}{250}, \frac{45}{250}\right) \tag{4.8}$$

is such a pair. With this pair the first term on the right hand side becomes

$$\left(F^{92} M^{171} N^{263}\right)^{\frac{1}{342}}. \tag{4.9}$$

Now let us write

$$S_{\alpha,\beta,\gamma}(x; m, N) = \sum_{\substack{M<m\leq 2M \\ N<n\leq 2N \\ m^{\alpha+\beta} n^{\gamma} \leq x \\ m>n}} \psi\left(\left(\frac{x}{m^{\beta} n^{\gamma}}\right)^{1/\alpha}\right).$$

With $z = x^{1/\alpha}$, $\rho = \beta/\alpha$, $\sigma = \gamma/\alpha$ we have, from Lemma 4.1 with the exponent pair (4.8), that

$$S_{\alpha,\beta,\gamma}(x; M, N) << \left(F^{92} M^{171} N^{263}\right)^{\frac{1}{342}} + F^{\frac{1}{4}} M^{\frac{1}{4}} N + F^{-\frac{1}{2}} M N,$$

where

$$F = (x\, M^{-\beta}\, N^{-\gamma})^{1/\alpha} \quad .$$

Let (α,β,γ) be any permutation of $(3,4,5)$. Then

$$M^4\, N^8 \ll (MN)^6 \ll M^{\alpha+\beta}\, N^{\gamma} \ll x$$

and so

$$F^{\frac{1}{4}}\, M^{\frac{1}{4}}\, N = \left\{x(M^4\, N^8)^{\alpha/2}(M^{\alpha+\beta}\, N^{\gamma})^{-1}\right\}^{\frac{1}{4\alpha}} \ll x^{\frac{1}{8}} \; ,$$

$$F^{-\frac{1}{2}} M\, N = \left\{x^{-1}(M^4\, N^8)^{\alpha/4}\, M^{\alpha+\beta}\, N^{\gamma}\right\}^{\frac{1}{2\alpha}} \ll x^{\frac{1}{8}}$$

and

$$\left(F^{92}\, M^{171}\, N^{263}\right)^{\frac{1}{342}} = \left\{x^{92}(MN)^{263\alpha}(M^{\alpha+\beta}\, N^{\gamma})^{-92}\right\}^{\frac{1}{342\alpha}} \ll x^{\frac{263}{2052}} .$$

Therefore

$$S_{\alpha,\beta,\gamma}(x;\, M,\, N) \ll x^{\frac{263}{2052}}$$

and so, from (4.3),

$$S_{\alpha,\beta,\gamma}(x) \ll x^{\frac{263}{2052}} \log^2 x \; .$$

The required result now follows from (4.6) and Theorem 1.

4.5 PROOF OF THEOREM 3.

We see from Theorem 1 that we need deal only with $\Delta_k^*(x)$. Suppose first that $4 \leqslant k \leqslant 12$. Let x be half an odd positive integer and $\varepsilon > 0$. We put

$$c = \frac{1}{k} + \varepsilon\,, \qquad T = x^M \qquad (M > 1 + c). \tag{5.1}$$

From (2.3) and (2.8) and the usual method (see, for example, Titchmarsh [36], p. 53.) of finding the partial sum of the coefficients of a Dirichlet series we have that

$$S_k(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{H_k(s)\, x^s}{s}\, ds + R(x,T) \tag{5.2}$$

where

$$|R(x,T)| \leqslant \frac{1}{\pi} \sum_{n=1}^{\infty} \tau_k(n) \left(\frac{x}{n}\right)^c \min\left(1, \frac{1}{T\left|\log \frac{x}{n}\right|}\right)$$

$$\ll \frac{x^{c+1}}{T} \ll 1\,, \tag{5.3}$$

by (5.1). We move the line of integration from $\sigma = c$ to $\sigma = 1/2k$ passing through the k simple poles of the integrand at

$$s = \frac{1}{k}\,,\ \frac{1}{k+1}\,,\ \ldots\,,\ \frac{1}{2k-1}\ .$$

The sum of the residues at these poles is

$$\sum_{k \leqslant r < 2k} A^*_{kr}\, x^{\frac{1}{r}}$$

where A^*_{kr} is defined in (1.4). From the definition of $\Delta^*_k(x)$ together with (5.2) and (5.3) we now have, from the residue theorem, that

$$\Delta^*_k(x) = \frac{1}{2\pi i}\left(I_1 + I_2 + I_3\right) + O(1) \tag{5.4}$$

where I_1, I_2, I_3 are the integrals of $H_k(s)\, x^s/s$ along the lines joining the points

$$c - iT\ , \qquad \frac{1}{2k} - iT\ , \qquad \frac{1}{2k} + iT\ , \qquad c + iT$$

in the respective order.

We first estimate I_3. Here we have

$$s = \sigma + iT\ , \qquad \frac{1}{2k} < \sigma < c\ .$$

From the well known estimates (see, for example, Titchmarsh [36], p. 99.):

$$\zeta(\tfrac{1}{2} + it) \ll t^{\frac{1}{6}}\ , \qquad \zeta(1 + it) \ll t^{\varepsilon}\ , \qquad (t > 1)$$

we have that

$$\zeta(ns) \ll T^{\frac{1}{3}(1-n\sigma)+\varepsilon}\ , \qquad \frac{1}{2n} \leq \sigma \leq \frac{1}{n}\ . \tag{5.5}$$

Let us put

$$I_3(n) = \int_{\frac{1}{n+1} + iT}^{\frac{1}{n} + iT} \frac{H_k(s)\, x^s}{s}\, ds\ , \qquad k \leq n < 2k$$

so that

$$I_3 = \sum_{k \leq n < 2k} I_3(n) + O(1)\ , \tag{5.6}$$

since

$$\int_{\frac{1}{k} + iT}^{c + iT} \frac{H_k(s)\, x^s}{s}\, ds \ll \frac{x^c}{T} \ll 1.$$

Let

$$\beta_n = \frac{1}{3} \sum_{r=k}^{n} \left(1 - \frac{r}{n}\right) = \frac{1}{6n} (n-k)(n-k+1) \quad , \tag{5.7}$$

and we shall prove that, for $k \leq n < 2k$,

$$I_3(n) << \frac{x^{\frac{1}{n} + \varepsilon}}{T^{1-\beta_n}} + \frac{x^{\frac{1}{n+1} + \varepsilon}}{T^{1-\beta_{n+1}}} \quad . \tag{5.8}$$

For r satisfying $n < r < 2k$ we have

$$\zeta(rs) << T^{\varepsilon} \quad , \qquad \frac{1}{n+1} < \sigma < \frac{1}{n}$$

and therefore, from (5.5),

$$H_k(s) << T^{\frac{1}{3} \sum_{r=k}^{n} (1-r\sigma)+k\varepsilon} \quad , \qquad \frac{1}{n+1} < \sigma < \frac{1}{n} \quad .$$

It follows that

$$I_3(n) << \int_{\frac{1}{n+1}}^{\frac{1}{n}} T^{\frac{1}{3} \sum_{r=k}^{n} (1-r\sigma)+k\varepsilon-1} x^{\sigma} \, d\sigma$$

$$<< \frac{x^{\frac{1}{n}} T^{k\varepsilon}}{T^{1-\beta_n}} + \frac{x^{\frac{1}{n+1}} T^{k\varepsilon}}{T^{1-\beta_{n+1}}}$$

since the integrand is maximum at $\sigma = 1/n$ or $\sigma = 1/(n+1)$, and

$$\frac{1}{3} \sum_{r=k}^{n} \left(1 - \frac{r}{n+1}\right) = \beta_{n+1} \quad .$$

Since $T^k = x^{Mk\varepsilon}$ by (5.1) we see that (5.8) is proved if we replace ε by ε/Mk.

From (5.6) and (5.8) we now have that

$$I_3 \ll \sum_{k \leq n < 2k} \frac{x^{\frac{1}{n}+\varepsilon}}{T^{1-\beta_n}} + 1 .$$

Similarly we have

$$I_1 \ll \sum_{k \leq n < 2k} \frac{x^{\frac{1}{n}+\varepsilon}}{T^{1-\beta_n}} + 1 .$$

We next deal with I_2. Let us write

$$f(s) = \prod_{k<n<2k} \zeta(ns)$$

so that

$$I_2 = \int_{\frac{1}{2k}-iT}^{\frac{1}{2k}+iT} \frac{\zeta(ks)f(s)x^s}{s}\,ds$$

$$\ll x^{\frac{1}{2k}} \int_1^T \frac{|\zeta(\frac{1}{2}+ikt)f(\frac{1}{2k}+it)|}{t}\,dt + x^{\frac{1}{2k}} ,$$

since the integral along $\frac{1}{2k} - i$ and $\frac{1}{2k} + i$ is bounded. Applying the Cauchy-Schwarz inequality to the integral we have that

$$I_2 \ll x^{\frac{1}{2k}} \left\{ \int_1^T \frac{|\zeta(\frac{1}{2}+ikt)|^2}{t}\,dt \int_1^T \frac{|f(\frac{1}{2k}+it)|^2}{t}\,dt \right\}^{\frac{1}{2}} + x^{\frac{1}{2k}} .$$

From Titchmarsh [36], Theorems 7.3 and 7.1 we have

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, dt \ll T \log T ,$$

and

$$\int_0^T |f(\tfrac{1}{2k} + it)|^2 \, dt \ll T .$$

Let R be the integer satisfying $2^{R-1} < T \leq 2^R$. Then

$$\int_1^T \frac{|\zeta(\frac{1}{2} + ikt)|^2}{t} \, dt \leq \sum_{r=1}^{R} 2^{1-r} \int_{2^{r-1}}^{2^r} |\zeta(\tfrac{1}{2} + ikt)|^2 \, dt$$

$$\ll \sum_{r=1}^{R} 2^{1-r} . \, 2^r \log 2^r \ll \sum_{r=1}^{R} r$$

$$\ll R^2 \ll \log^2 T ,$$

and similarly

$$\int_1^T \frac{|f(\frac{1}{2k} + it)|^2}{t} \, dt \ll \log T .$$

Therefore we arrive at

$$I_2 \ll x^{\frac{1}{2k}} \log^{\frac{3}{2}} T .$$

Returning to (5.4) we now have that

$$\Delta_k^*(x) \ll x^{\frac{1}{2k}} \log^{\frac{3}{2}} T + \sum_{k \leq n < 2k} \frac{x^{\frac{1}{n} + \varepsilon}}{T^{1-\beta_n}} + 1 .$$

Since $4 \leq k \leq 12$ we see from (5.7) that $\beta_n < 1$ for $k \leq n < 2k$.

With M sufficiently large in (5.1), the sum on the right hand side is bounded and therefore

$$\Delta_k^*(x) << x^{\frac{1}{2k}} \log^{\frac{3}{2}} x \ , \qquad 4 \leqslant k \leqslant 12.$$

Suppose now that $k \geqslant 13$. From (4.7) we see that $\beta_n < 1$ provided that $k \leqslant n < k+u$ where u is given in the theorem. We now consider the integral in (5.2) and move the line of integration from $\sigma = c$ to $\sigma = 1/(k+u)$ only. The same argument can be applied, except that we now have that

$$I_1 + I_3 << \sum_{k\leqslant n<k+u} \frac{x^{\frac{1}{n}+\varepsilon}}{T^{1-\beta_n}} + x^{\frac{1}{k+u}+\varepsilon}$$

and

$$I_2 = \int_{\frac{1}{k+u}-iT}^{\frac{1}{k+u}+iT} \frac{H_k(s)\, x^s}{s}\, ds$$

$$<< x^{\frac{1}{k+u}} \int_1^T \frac{|H_k(\frac{1}{k+u}+it)|}{t}\, dt + x^{\frac{1}{k+u}}$$

$$<< x^{\frac{1}{k+u}} \log T \ .$$

From these estimates we deduce that

$$\Delta_k^*(x) << x^{\frac{1}{k+u}+\varepsilon} \ , \qquad k \geqslant 13$$

and hence

$$\rho_k^* \leqslant \frac{1}{k+u} \ , \qquad k \geqslant 13 \ .$$

If we apply deeper theorems concerning the order of $\zeta(s)$ on the critical line $\sigma = \frac{1}{2}$ we can improve on our result for $k \geqslant 13$ slightly. In fact, if $0 < \lambda < 1/6$ and

$$\zeta(\tfrac{1}{2} + it) << t^{\lambda}, \qquad t \to \infty,$$

then we can take

$$\beta_n = 2\lambda \sum_{r=k}^{n} \left(1 - \frac{r}{n}\right)$$

and now the condition $\beta_n < 1$ will hold for $n < k + u'$ where $u' > u$. In particular, assuming the Lindelöf hypothesis, namely that

$$\zeta(\tfrac{1}{2} + it) << t^{\varepsilon}, \qquad t \to \infty,$$

for all $\varepsilon > 0$, we have that

$$\Delta_k^*(x) << x^{\frac{1}{2k}} \log^{\frac{3}{2}} x, \qquad k \geqslant 13.$$

The theorem now follows from these results and Theorem 1.

4.6 A PROBLEM OF ANALYTIC CONTINUATION.

For $k \geqslant 3$ the line $\sigma = 0$ is a natural boundary to the function $J_k(s)$ defined in (2.4). The proof requires only a small modification of the proof of the main theorem in Estermann's paper [8]. We shall restrict ourselves to the case $k = 3$.

Let us write, for $\sigma > 1$,

$$g(s) = \prod_p \left(1 - p^{-s} + p^{-3s}\right) \tag{6.1}$$

so that, by (2.1), (2.3) and (2.5),

$$J_3(s) = \frac{\zeta(s)\, g(s)}{\zeta(3s)\zeta(4s)\zeta(5s)} .$$

Since $\zeta(s)$, $\zeta(3s)$, $\zeta(4s)$ and $\zeta(5s)$ are meromorphic in the whole s-plane it remains to show that $\sigma = 0$ is a natural boundary to $g(s)$.

Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of the equation

$$\lambda^3 - \lambda^2 + 1 = 0$$

and let

$$\alpha = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|) > 1 .$$

We recall that Estermann proved that

$$\ell(n) = \frac{1}{n} \sum_{ab=n} \mu(a)\,(\lambda_1^b + \lambda_2^b + \lambda_3^b)$$

is an integer which satisfies

$$1 - x + x^3 = \prod_{n=1}^{\infty} (1 - x^n)^{\ell(n)} , \qquad |x| < \frac{1}{\alpha} , \tag{6.2}$$

and it is clear that

$$|\ell(n)| \leqslant 3\alpha^n , \qquad n = 1,2,3,\ldots .$$

Let $0 < \delta < 1$, $M > \alpha^{1/\delta}$. For $p > M$ and $\sigma \geqslant \delta$ we have

$$\left|\frac{\ell(n)}{p^{n\delta}}\right| \leqslant 3 \left(\frac{\alpha}{p^\sigma}\right)^n$$

and since

$$\frac{\alpha}{p^{\sigma}} < \frac{\alpha}{M^{\delta}} < 1$$

it follows that

$$\sum_{n=1}^{\infty} \left|\frac{\ell(n)}{p^{n\delta}}\right| \leq \frac{3\alpha}{p^{\sigma}} \cdot \frac{1}{1 - \alpha p^{-\delta}} < \frac{3\alpha}{1 - \alpha M^{-\delta}} \cdot \frac{1}{p^{\sigma}} .$$

Therefore

$$\sum_{p>M} \sum_{n=1}^{\infty} \left|\frac{\ell(n)}{p^{ns}}\right| < \infty , \qquad \sigma > 1$$

and so

$$\prod_{p>M} \prod_{n=1}^{\infty} (1 - p^{-ns})^{\ell(n)} = \prod_{n=1}^{\infty} \left\{\prod_{p>M} (1 - p^{-ns})\right\}^{\ell(n)} \qquad \sigma > 1 .$$

From (6.2) we now have, for $\sigma > 1$,

$$\prod_{p>M} (1 - p^{-s} + p^{-3s}) = \prod_{n=1}^{\infty} \left\{\zeta_M(ns)\right\}^{-\ell(n)}$$

where

$$\zeta_M(ns) = \prod_{p>M} (1 - p^{-ns})^{-1} = \zeta(ns) \prod_{p\leq M} (1 - p^{-ns}) .$$

From (6.1) we now have, for $\sigma > 1$,

$$g(s) = \prod_{p\leq M} (1 - p^{-s} + p^{-3s}). \prod_{n=1}^{\infty} \left\{\zeta_M(ns)\right\}^{-\ell(n)}$$

$$= g_1(s)\, g_2(s)\, g_3(s) ,$$

say, where

$$g_1(s) = \prod_{p \leq M} (1 - p^{-s} + p^{-3s}) \quad ,$$

$$g_2(s) = \prod_{n \leq [1/\delta]} \left\{\zeta_M(ns)\right\}^{-\ell(n)} \quad ,$$

$$g_3(s) = \prod_{n > [1/\delta]} \left\{\zeta_M(ns)\right\}^{-\ell(n)} \quad .$$

Now $g_1(s)$ is an entire function, and $g_2(s)$ is meromorphic in the whole s-plane. We next show that the product for $g_3(s)$ converges uniformly in $\sigma \geq \delta$. In fact, for

$$n \geq \left[\frac{1}{\delta}\right] + 1 \quad \text{and} \quad \sigma \geq \delta \quad ,$$

we have

$$|\zeta_M(ns) - 1| \leq \sum_{m=M+1}^{\infty} m^{-n\delta} < \frac{1}{n\delta - 1} \cdot \frac{M}{M^{n\delta}}$$

and

$$\frac{|\ell(n)|}{M^{n\delta}} \leq 3 \left(\frac{\alpha}{M^{\delta}}\right)^n \quad , \qquad \frac{\alpha}{M^{\delta}} < 1$$

so that

$$\sum_{n > [1/\delta]}^{\infty} \frac{|\ell(n)| M}{(n\delta - 1) M^{n\delta}} < \infty .$$

Therefore the series

$$\sum_{n > [1/\delta]}^{\infty} |\ell(n)| |\zeta_M(n\delta) - 1|$$

converges uniformly in $\sigma \geq \delta$, and hence the product for $g_3(s)$

converges uniformly in $\sigma \geq \delta$. Therefore $g_3(s)$ is meromorphic in $\sigma > \delta$ and since δ is arbitrary it follows that $g(s)$ has a meromorphic continuation into $\sigma > 0$.

We next show that the zeros of $g(s)$ are dense in the line $\sigma = 0$. Let $\varepsilon > 0$. We show that $g(s)$ has a zero in the square

$$0 < \sigma < \varepsilon \;, \qquad u < t < u+\varepsilon \tag{6.3}$$

where we can assume that $u > 0$. We shall also assume that $\alpha = |\lambda_1|$ so that we can write

$$\lambda_1 = e^{\beta+i\gamma} \;, \qquad \alpha = e^{\beta} \;, \qquad \beta > 0 \;.$$

For $x > 0$ we shall denote by $\overline{w}(x)$ the number of primes not exceeding x, and by $N(x)$ the number of zeros of $\zeta(s)$ with $0 < t < x$. We can choose V so large that

$$0 < \frac{1}{V} < \varepsilon \;, \qquad \frac{2\pi}{V\beta} < \varepsilon \;,$$

$$\overline{w}(e^{2V\beta}) - \overline{w}(e^{V\beta}) > e^{V\beta} \tag{6.4}$$

and

$$e^{V\beta} > 2V\, N(2V(u + \varepsilon)). \tag{6.5}$$

The last two inequalities being possible because, as $x \to \infty$, we have the well known estimates

$$\overline{w}(x) \sim \frac{x}{\log x} \;, \qquad N(x) \ll x \log x.$$

We note that the rectangle

$$\frac{1}{2V} \leq \sigma < \frac{1}{V} \;, \qquad u < t < u+\varepsilon \tag{6.6}$$

is contained in the square (6.3). Now take

$$\delta = \frac{1}{2V} , \qquad M = \left[\alpha^{\frac{1}{\delta}}\right] + 1 = \left[e^{2v\beta}\right] + 1.$$

Then $g_3(s)$ has no poles in the rectangle (6.6). Let Z denote the number of distinct zeros of $g_1(s)$ and let P denote the number of distinct poles of $g_2(s)$ in the rectangle (6.6), so that it suffices to prove that

$$Z > P . \tag{6.7}$$

Now $1 - p^{-s} + p^{-3s} = 0$ if $p^s = \lambda_1$, which is the case if

$$s \log p = \log \lambda_1 = \beta + i(\gamma + 2\pi m)$$

where m is any integer. Thus, if $s = \sigma + it$ where

$$\sigma = \frac{\beta}{\log p} , \qquad t = \frac{\gamma + 2\pi m}{\log p} , \qquad p \leqslant M \tag{6.8}$$

then s is a zero of $g_1(s)$. If, moreover,

$$e^{V\beta} < p \leqslant e^{2V\beta}$$

so that

$$\frac{1}{2V} \leqslant \frac{\beta}{\log p} < \frac{1}{V} , \qquad \frac{2\pi}{\log p} < \varepsilon$$

which means that, for each such prime p, at least one of the numbers s given in (6.8) is in the rectangle (6.6) and so

$$Z \geqslant \overline{w}\left(e^{2V\beta}\right) - \overline{w}\left(e^{V\beta}\right) . \tag{6.9}$$

On the other hand, each pole of $g_2(s)$ must be a zero of $\zeta_M(s), \zeta_M(2s), \ldots, \zeta_M(2Vs)$ which have the same zeros in $\sigma > 0$

as $\zeta(s)$, $\zeta(2s)$,..., $\zeta(2Vs)$. Hence

$$P \leqslant \sum_{n=1}^{2V} P_n$$

where P_n is the number of zeros of $\zeta(ns)$ in the rectangle (6.6), and this is the same as the number of zeros of $\zeta(s)$ in the rectangle

$$\frac{n}{2V} \leqslant \sigma < \frac{n}{V}, \qquad un < t < (u+\varepsilon)n$$

and so

$$P_n \leqslant N((u+\varepsilon)n) \leqslant N(2V(u+\varepsilon))$$

and hence

$$P \leqslant 2V\ N(2V(n+\varepsilon)) . \tag{6.10}$$

The required result (6.7) now follows from (6.4), (6.5), (6.9) and (6.10).

Chapter Five

The Distribution of Square-full Integers

5.1 INTRODUCTION.

In this chapter we study the finer distribution of the 2-full integers, or the square-full integers. We shall write $S(x)$ and $Q(s)$ for $S_2(x)$ and $Q_2(x)$ respectively. We recall from (4.1.3) and (4.1.7) that

$$S(x) = A_{22}^{*} x^{\frac{1}{2}} + A_{23}^{*} x^{\frac{1}{3}} + \Delta_2^{*}(x) , \tag{1.1}$$

and

$$Q(x) = A_{22} x^{\frac{1}{2}} + A_{23} x^{\frac{1}{3}} + \Delta_2(x) , \tag{1.2}$$

and that ρ_2^{*} and ρ_2 are the infima of the sets of exponents associated with the error terms $\Delta_2^{*}(x)$ and $\Delta_2(x)$ respectively. We also have, from (4.1.12), that

$$\rho_2^{*} \leqslant \frac{2}{15} \tag{1.3}$$

and, by Theorem 4.1,

$$\rho_2 \leqslant \frac{1}{6} . \tag{1.4}$$

As we remarked in Chapter Four, (1.3) is due to Richert and it is the consequence of a particular case of his general result on the estimation of the sum

$$R(x;\, \alpha,\, \beta,\gamma) = \sum_{n \leqslant x^{\alpha}} \psi\left(\frac{x^{\beta}}{n^{\gamma}}\right) ,$$

using the method of exponent pairs. In section 2 we shall choose a more suitable exponent pair for our particular problem, and prove that

$$\rho_2^* \leqslant \frac{12}{91} . \tag{1.5}$$

If $\rho_2 < \theta < \frac{1}{2}$, then it is easy to deduce from (1.2) that, as $x \to \infty$,

$$Q\left(x + x^{\frac{1}{2} + \theta}\right) - Q(x) \sim \frac{1}{2} A_{22} x^\theta . \tag{1.6}$$

The question is whether there exists a $\theta \leqslant \rho_2$ such that (1.6) is true. Let θ_o be the infinum of all such θ so that, by (1.4),

$$\theta_o \leqslant \frac{1}{6} = 0.166 \ldots . \tag{1.7}$$

In section 3 we shall prove:

<u>Theorem 1.</u> <u>We have</u>

$$\theta_o \leqslant \frac{1 + \rho_2^*}{9 - 12\rho_2^*} .$$

From (1.5) and Theorem 1 we deduce that

$$\theta_o \leqslant \frac{103}{675} = 0.1529 \ldots ,$$

which is an improvement on (1.7).

Let (q_n) denote the sequence of square-full integers. This sequence contains all the perfect squares together with square-full integers which are not squares, these being numbers of the form

$$q = a^2 b^3 , \qquad \mu^2(b) = 1 , \qquad b > 1 .$$

There are $\left[x^{1/2}\right]$ squares not exceeding x, and since

$$A_{22} = \frac{\zeta(\frac{3}{2})}{\zeta(3)} = 2.1732\ldots > 2$$

we see from (1.2) that the squares form a minority in the sequence (q_n). As a consequence of this the problem of whether there are infinitely many pairs of consecutive squares in the sequence (q_n) is not trivial. Here we prove much more.

Theorem 2. Let $f(n)$ denote the number of square-full integers in the interval $n^2 < q < (n+1)^2$, and let

$$F_m = \{n : f(n) = m\}, \quad m = 0,1,2,\ldots .$$

Then each F_m has positive asymptotic density d_m given by

$$d_m = \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{(m+\ell)!}{m!\,\ell!} c_{m+\ell}$$

where

$$c_0 = 1, \qquad c_r = \sum_{1<b_1<\ldots<b_r} \frac{\mu^2(b_1)\ldots\mu^2(b_r)}{(b_1\ldots b_r)^{3/2}}, \qquad r = 1,2,\ldots .$$

In particular, since

$$d_0 = \sum_{\ell=0}^{\infty} (-1)^{\ell} c_{\ell} > 0$$

(see the calculations in section 5.5), it follows that (q_n) does indeed contain infinitely many pairs of consecutive terms that are both squares.

Let $a(n)$ denote the number of non-isomorphic Abelian groups of order n. Kendall and Rankin [14] showed that each of the sets $\{n : a(n) = m\}$ $m = 0,1,2,\ldots$, has asymptotic density P_m, and they call the sequence (P_m) the asymptotic frequency distribution of $a(n)$. They remarked that $a(n)$ is a rare example of an integer-valued function with finite mean value for which the asymptotic frequency distribution (P_m) can be calculated explicity and satisfies

$$\sum_{m=0}^{\infty} P_m = 1, \qquad \sum_{m=0}^{\infty} m\, P_m = \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} a(n).$$

We remark that the function $f(n)$ in Theorem 2 is another such example. (Note that $d_m > 0$ for all m whereas $P_o = P_{13} = 0$.)

Since the equation $x^2 = 8y^2 + 1$ has infinitely many solutions in integers we see at once that

$$\liminf_{n\to\infty} (q_{n+1} - q_n) = 1 .$$

In our final section we deduce from (1.2) and $d_o > 0$ that

$$\limsup_{n\to\infty} \frac{q_{n+1} - q_n}{2n} = \frac{1}{A_{22}} . \tag{1.8}$$

5.2 PROOF OF (1.5).

First we have

$$\Delta_2^*(x) = - R(x; \tfrac{1}{5}, \tfrac{1}{2}, \tfrac{3}{2}) - R(x\, ; \tfrac{1}{5}, \tfrac{1}{3}, \tfrac{2}{3}) + O(1) \tag{2.1}$$

where, for positive α, β, γ,

$$R(x; \alpha, \beta, \gamma) = \sum_{n\leq x^{\alpha}} \psi \left(\frac{x^{\beta}}{n^{\gamma}}\right) . \tag{2.2}$$

As we pointed out in Chapter Four this is a particular case of a general result due to Richert [27]. We see from (2.1) and (2.2) that trivially $\Delta_2^*(x) << x^{1/5}$. Since

$$\psi(x) = x - [x] - \frac{1}{2} = - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n}$$

whenever x is not an integer, the sum in (2.2) can be transformed into a trigonometric sum. If we apply Vinogradov's method of estimating trigonometric sums (see, for example, Gelfond and Linnik [9] p.   ) we can prove that $\Delta_2^*(x) << x^{1/7}$, giving $\rho_2^* \leqslant 1/7$. Although this is inferior to (1.3), it is nevertheless a significant result in the sense that it establishes the inequality (4.1.11) when k = 2.

Van der Corput's method of estimating trigonometric sums has been developed into a delicate theory of exponent pairs due to van der Corput [38], Phillips [24], and Rankin [26] . The rather complicated definition of an exponent pair is given in [24] and [26]. By means of this theory Richert [27] has prove the following:

<u>Lemma 2.1.</u> (Richert [27], Lemma 8).

<u>Let</u> α, β, γ <u>be positive constants and let</u> (k, ℓ) <u>be an exponent pair with</u> $k > 0$. <u>Then, as</u> $x \to \infty$,

$$R(x; \alpha, \beta, \gamma) << x^{\alpha - \frac{1}{2}(\beta - \alpha\gamma)} + \begin{cases} x^{\frac{\alpha\ell + (\beta - \alpha\gamma)k}{k+1}} & \text{if } \ell > \gamma k , \\ x^{\frac{\beta k}{k+1}} \log x & \text{if } \ell = \gamma k , \\ x^{\frac{\beta k}{1 + (\gamma+1)k - \ell}} & \text{if } \ell < \gamma k . \end{cases}$$

Let (k,ℓ) be an exponent pair such that $2\ell = 3k$. We see from Lemma 2.1 that

$$R(x; \tfrac{1}{5}, \tfrac{1}{2}, \tfrac{3}{2}) \ll x^{\frac{1}{10}} + x^{\frac{k}{2(k+1)}} \log x ,$$

and

$$R(x; \tfrac{1}{5}, \tfrac{1}{3}, \tfrac{2}{3}) \ll x^{\frac{1}{10}} + x^{\frac{k}{2(k+1)}} .$$

By the definition of an exponent pair we must have $\ell \geqslant \frac{1}{2}$ so that $k \geqslant 1/3$ and hence $k/2(k+1) \geqslant 1/8$. It now follows from (2.1) that

$$\Delta_2^*(x) \ll x^{\frac{k}{2(k+1)}} \log x$$

giving

$$\rho_2^* \leqslant \frac{k}{2(k+1)} . \tag{2.3}$$

By an application of the A-process Phillips [24] showed that if (k,ℓ) is an exponent pair, then so is $(k_1,\ell_1) = A(k,\ell)$ where

$$k_1 = \frac{k}{2(k+1)} , \qquad \ell_1 = \frac{1}{2} + \frac{\ell}{2(k+1)} .$$

By applying the B-process he showed that if (k,ℓ) is a exponent pair, then so is $(k_1,\ell_1) = B(k,\ell)$ where

$$k_1 = \ell - \frac{1}{2} , \qquad \ell_1 = k + \frac{1}{2} .$$

Rankin [26] showed that there is also a convexity process which asserts that if (k_1,ℓ_1) and (k_2,ℓ_2) are exponent pairs, then so is

$$(k,\ell) = C\left\{(k_1,\ell_1), (k_2,\ell_2); t\right\}$$

where

$$k = t\,k_1 + (1-t)k_2\,, \qquad \ell = t\,\ell_1 + (1-t)\ell_2\,, \qquad (0 \leqslant t \leqslant 1).$$

Now let $(\eta, \frac{1}{2} + \eta)$ be an exponent pair, and consider the two exponent pairs

$$(k_1,\ell_1) = BA(\eta, \tfrac{1}{2} + \eta) = \left(\frac{\eta + \frac{1}{2}}{2\eta + 2}\,,\ \frac{2\eta + 1}{2\eta + 2}\right),$$

$$(k_2,\ell_2) = BA^2(\eta, \tfrac{1}{2} + \eta) = \left(\frac{2\eta + \frac{3}{2}}{6\eta + 4}\,,\ \frac{4\eta + 2}{6\eta + 4}\right).$$

We can apply the convexity process C to these two pairs to give an exponent pair (k,ℓ) such that $2\ell = 3k$. Specifically, let

$$t = \frac{1 - 3\eta - 4\eta^2}{3 + 4\eta + 2\eta^2}\,.$$

Then we have the exponent pair

$$C\Big\{(k_1,\ell_1),(k_2,\ell_2);\ t\Big\} = \left(\frac{2\eta + 1}{2\eta^2 + 4\eta + 3}\,,\ \frac{3}{2} \times \frac{2\eta + 1}{2\eta^2 + 4\eta + 3}\right).$$

It now follows from (2.3) that

$$\rho_2^* \leqslant \frac{2\eta + 1}{4(\eta+1)(\eta+2)} \tag{2.4}$$

where η is any number such that $(\eta, \frac{1}{2} + \eta)$ is an exponent pair.

Now, as an immediate consequence of the definition of an exponent pair, (0,1) is an exponent pair so that we have the exponent pair

$$\left(\frac{1}{6}\,,\ \frac{2}{3}\right) = AB(0,1)$$

and so we can take $\eta = 1/6$ in (2.4) giving our required result (1.5).

With more tedious calculations we can improve on (1.5) slightly. For example, we can consider

$$\left(\frac{11}{82}, \frac{57}{82}\right) = ABA^2\left(\frac{1}{6}, \frac{2}{3}\right)$$

$$\left(\frac{16}{82}, \frac{52}{82}\right) = B\left(\frac{11}{82}, \frac{57}{82}\right)$$

$$\left(\frac{27}{164}, \frac{109}{164}\right) = C\left\{\left(\frac{11}{82}, \frac{57}{82}\right), \left(\frac{16}{82}, \frac{52}{82}\right); \frac{1}{2}\right\}$$

so that we can take $\eta = 27/164$ giving

$$\rho_2^* \leq \frac{8938}{67805} < \frac{12}{91}.$$

Let η_o be the infimum of all η such that $(\eta, \frac{1}{2} + \eta)$ is an exponent pair so that we have

$$\rho_2^* \leq \frac{2\eta_o + 1}{4(\eta_o+1)(\eta_o+2)}.$$

From Rankin's calculations [26] we have

$$\eta_o = 0.1645106784 \ldots$$

giving

$$\rho_2^* \leq 0.13181619 \ldots .$$

5.3 PROOF OF THEOREM 1.

Let ρ and θ satisfy

$$\rho_2^* < \rho < \frac{1}{6}, \qquad 0 < \theta < \frac{1}{2}, \tag{3.1}$$

and for $x > 1$ we let

$$h = x^{\frac{1}{2} + \theta}, \qquad \log x < t < x^{\frac{1}{6}}. \tag{3.2}$$

First we have, from (4.1.1),

$$Q(x) = \sum_{q \leq x} \sum_{a^2b^3=q} \mu^2(b) = \sum_{q \leq x} \sum_{a^2b^3=q} \sum_{c^2d=b} \mu(c) = \sum_{a^2d^3c^6 \leq x} \mu(c),$$

so that

$$Q(x+h) - Q(x) = \sum_{x<a^2b^3c^6 \leq x+h} \mu(c)$$

$$= \sum_{\substack{x<a^2b^3c^6 \leq x+h \\ c \leq t}} \mu(c) + \sum_{\substack{x<a^2b^3c^6 \leq x+h \\ c>t}} \mu(c)$$

$$= \sum\nolimits_1 + \sum\nolimits_2 \tag{3.3}$$

say. We have at once

$$\sum\nolimits_1 = \sum_{c \leq t} \mu(c) \sum_{\frac{x}{c^6}<a^2b^3 \leq \frac{x+h}{c^6}} 1$$

$$= \sum_{c \leq t} \mu(c) \left\{ S\left(\frac{x+h}{c^6}\right) - S\left(\frac{x}{c^6}\right) \right\}.$$

Now, as $x \to \infty$, we have

$$(x+h)^{\frac{1}{2}} - x^{\frac{1}{2}} = x^{\theta} + O\left(x^{2\theta - \frac{1}{2}}\right), \tag{3.4}$$

$$(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}} \ll x^{\theta - \frac{1}{6}}, \tag{3.5}$$

and

$$\sum_{c \leq t} \left(\frac{x}{c^6}\right)^\rho << x^\rho \ t^{1-6\rho}$$

so that, from (1.1) and (4.1.6), we arrive at

$$\sum\nolimits_1 = \left(\frac{1}{2} A_{22} + o(1)\right) x^\theta + O\left(x^\rho \ t^{1-6\rho}\right) . \tag{3.6}$$

Next we put

$$T(x,t) = \sum_{\substack{a^2 b^3 c^6 \leq x \\ c > t}} 1 \tag{3.7}$$

so that

$$\left|\sum\nolimits_2\right| \leq T(x+h, \ t) - T(x,t)$$

and it now follows from (3.3) and (3.6) that

$$\left|Q(x+h) - Q(x) - \left(\frac{1}{2} A_{22} + o(1)\right) x^\theta\right| \leq T(x+h,t) - T(x,t) + O\left(x^\rho t^{1-6\rho}\right) . \tag{3.8}$$

We first use the following crude method to estimate $T(x+h, \ t) - T(x,t)$. Let u be a number satisfying

$$ut^5 > h . \tag{3.9}$$

Corresponding to each pair of numbers a, b there are at most u numbers c satisfying $c > t$ and $x < a^2 b^3 c^6 \leq x+h$, since

$$a^2 b^3 (c+u)^6 > a^2 b^3 c^6 + uc^5 > x + ut^5 > x + h .$$

We conclude that

$$T(x+h,t) - T(x,t) \leq \sum_{a^2 b^3 \leq 2xt^{-6}} u = u \ S(2xt^{-6}) << ux^{\frac{1}{2}} \ t^{-3} .$$

We now set

$$t = x^{\frac{1}{8}} \log x , \qquad u = x^{\theta - \frac{1}{8}}$$

so that (3.9) holds by (3.2), and that

$$T(x+h,\ t) - T(x,t) = O\left(x^{\theta}(\log x)^{-3}\right) = o(x^{\theta}) \ .$$

From (3.8) we now have that

$$Q(x+h) - Q(x) = \left(\frac{1}{2} A_{22} + o(1)\right)x^{\theta} + O\left(x^{\frac{1+2\rho}{8}} (\log x)^{1-6\rho}\right).$$

Consequently if, in (3.1), θ further satisfies

$$\frac{1 + 2\rho}{8} < \theta < \frac{1}{2} ,$$

then the asymptotic formula (1.6) holds. This proves that

$$\theta_o \leqslant \frac{1 + 2\rho_2^*}{8}$$

and on applying (1.5) we arrive at

$$\theta_o \leqslant \frac{115}{728} = 0.1579... \tag{3.10}$$

which is already an improvement on (1.7).

We next make a more careful estimate for $T(x+h),t) - T(x,t)$. We recall from section 4.2 that

$$\tau_2(n) = \sum_{a^2 b^3 = n} 1$$

so that, from (3.7),

$$T(x,t) = \sum_{\substack{nc^6 \leq x \\ c>t}} \tau_2(n) = \sum_{n \leq xt^{-6}} \tau_2(n) \sum_{t<c\leq (xn^{-1})^{1/6}} 1$$

$$= \sum_{n\leq xt^{-6}} \tau_2(n)\left\{\left[\left(\frac{x}{n}\right)^{\frac{1}{6}}\right] - [t]\right\}$$

$$= \sum_{n\leq xt^{-6}} \tau_2(n)\left\{\left(\frac{x}{n}\right)^{\frac{1}{6}} - \psi\left(\left(\frac{x}{n}\right)^{\frac{1}{6}}\right)\right\} - tS(xt^{-6}) \;, \tag{3.11}$$

provided that $\psi(t) = 0$, and this we may assume by taking t to be half an odd integer. Now, by (1.1),

$$tS(xt^{-6}) = A_{22}^{*}\, x^{\frac{1}{2}}\, t^{-2} + A_{23}^{*}\, x^{\frac{1}{3}}\, t^{-1} + t\Delta_2^{*}(xt^{-6}) \;.$$

Similarly to the derivation of (1.1) itself we have that

$$\sum_{n\leq xt^{-6}} \tau_2(n)\left(\frac{x}{n}\right)^{\frac{1}{6}} = x^{\frac{1}{6}} \sum_{a^2b^3\leq xt^{-6}} \frac{1}{a^2b^3}$$

$$= \frac{3}{2}A_{22}^{*}\, x^{\frac{1}{2}}\, t^{-2} + 2A_{23}^{*}\, x^{\frac{1}{3}}\, t^{-1} + \zeta\left(\frac{1}{3}\right)\zeta\left(\frac{1}{2}\right)x^{\frac{1}{6}} + t\Delta_2^{*}(xt^{-6}) \;.$$

Let us write

$$U(x,t) = \sum_{n\leq xt^{-6}} \tau_2(n)\psi\left(\left(\frac{x}{n}\right)^{\frac{1}{6}}\right)$$

$$= \sum_{a^2b^3\leq xt^{-6}} \psi\left(\left(\frac{x}{a^2b^3}\right)^{\frac{1}{6}}\right) \;. \tag{3.12}$$

In view of (3.4) and (3.5), it now follows from (3.11) that

$$T(x+h),t) - T(x,t) = - U(x+h,t) + U(x,t) + O(x^{\rho} t^{1-6\rho}) + o(x^{\theta}) ,$$

and so, from (3.8), we now have

$$\left|Q(x+h) - Q(x) - \left(\frac{1}{2} A_{22} + o(1)\right)x^{\theta}\right| \leq \left|U(x+h,t) - U(x,t)\right| + O(x^{\rho}t^{1-6\rho}). \tag{3.13}$$

Using the trivial estimate in (3.12) we have

$$|U(x,t)| \leq \sum_{a^2b^3 \leq xt^{-6}} 1 = S(xt^{-6}) << x^{\frac{1}{2}} t^{-3} .$$

On setting

$$t = x^{\frac{1-2\rho}{8-12\rho}}$$

so that

$$x^{\frac{1}{2}} t^{-3} = x^{\rho} t^{1-6\rho} = x^{\frac{1}{8-12\rho}}$$

we arrive at

$$Q(x+h) - Q(x) = \left(\frac{1}{2} A_{22} + o(1)\right) x^{\theta} + O\left(x^{\frac{1}{8-12\rho}}\right).$$

Consequently if, in (3.1), θ satisfies also

$$\frac{1}{8-12\rho} < \theta < \frac{1}{2} ,$$

then the asymptotic formula (1.6) holds. This proves that

$$\theta_o \leq \frac{1}{8-12\rho_2^*} ,$$

and on applying (1.5) we arrive at

$$\theta_o \leq \frac{91}{584} = 0.1558 \ldots$$

which is an improvement on (3.10).

Our final improvement comes from a non-trivial estimate for $U(x,t)$ given by the following:

<u>Lemma 3.1.</u> <u>We have, as</u> $x \to \infty$,

$$U(x,t) << \left(x^{\frac{1}{2}} t^{-\frac{7}{2}} + x^{\frac{217}{855}} t^{-\frac{1072}{855}}\right) \log^2 x$$

<u>uniformly in</u> $t \leq x^{\frac{1}{6}}$.

From the lemma we see that, as $x \to \infty$,

$$U(x,t) << x^{\frac{1}{2}} t^{-\frac{7}{2}} \log^2 x, \qquad t \leq x^{\beta} , \tag{3.14}$$

where

$$\beta = \frac{421}{3841} = 0.1096 \ldots .$$

We finally set

$$t = x^{\frac{1-2\rho}{9-12\rho}}$$

so that

$$x^{\frac{1}{2}} t^{-\frac{7}{2}} = x^{\rho} t^{1-6\rho} = x^{\frac{1+\rho}{9-12\rho}} .$$

Since $\rho > \rho_2^* \geq \frac{1}{10}$ by (3.1) and (4.1.5) we see that the exponent of t is at most

$$\frac{4}{39} = 0.10256 \ldots < \beta$$

so that (3.14) is valid, and so from (3.13) we have that

$$Q(x+h) - Q(x) = \left(\frac{1}{2} A_{22} + o(1)\right) x^{\theta} + O\left(x^{\frac{1+\rho}{9-12\rho}} \log^2 x\right) .$$

Consequently if θ satisfies

$$\frac{1+\rho}{91-2\rho} < \theta < \frac{1}{2} ,$$

then the asymptotic formula (1.6) holds. This proves our required result that

$$\theta_o \leqslant \frac{1+\rho_2^*}{9-12\rho_2^*} \leqslant \frac{103}{675} = 0.1529 \ldots$$

by (1.5), subject to the proof of Lemma 3.1.

Let us write

$$U_1(x,t) = \sum_{\substack{m^2n^3 \leqslant xt^{-6} \\ m>n}} \psi\left(\left(\frac{x}{m^2n^3}\right)^{\frac{1}{6}}\right)$$

and

$$U_2(x,t) = \sum_{\substack{m^2n^3 \leqslant xt^{-6} \\ n>m}} \psi\left(\left(\frac{x}{m^2n^3}\right)^{\frac{1}{6}}\right)$$

so that, from (3.12),

$$U(x,t) = U_1(x,t) + U_2(x,t) + O\left(\left(xt^{-6}\right)^{\frac{1}{5}}\right) . \tag{3.15}$$

We shall apply Srinivasan's theorem, that is our Lemma 4.4.1, with the two dimensional exponent pair (4.4.8), to estimate $U_1(x,t)$ and $U_2(x,t)$.

Let $z = x^{1/6}$ and (ρ,σ) be either $\left(\frac{1}{2}, \frac{1}{3}\right)$ or $\left(\frac{1}{3}, \frac{1}{2}\right)$, and put

$$S_{\rho,\sigma}(x,t;\, M,N) = \sum_{\substack{M<m\leqslant 2M \\ N<n\leqslant 2N \\ m^\rho n^\sigma \leqslant zt^{-1} \\ m>n}} \psi\left(\frac{z}{m^\rho n^\sigma}\right) .$$

We have, from Lemma 4.4.1, that

$$S_{\rho,\sigma}(z,t;M,N) \ll (F^{92} M^{171} N^{263})^{\frac{1}{342}} + F^{\frac{1}{4}} M^{\frac{1}{4}} N + F^{-\frac{1}{2}} MN ,$$

where

$$F = z M^{-\rho} N^{-\sigma} .$$

Now

$$N^{\frac{5}{6}} \ll M^{\frac{1}{3}} N^{\frac{1}{2}} \ll M^{\rho} N^{\sigma} \ll z t^{-1} .$$

Therefore

$$F^{\frac{1}{4}} M^{\frac{1}{4}} N = z^{\frac{1}{4}} (M^{\rho}N^{\sigma})^{-\frac{1}{4}} M^{\frac{1}{4}} N$$

$$= z^{\frac{1}{4}} (M^{\rho}N^{\sigma})^{-\frac{1}{4}} \left(M^{\frac{1}{3}} N^{\frac{1}{2}}\right)^{\frac{3}{4}} \left(N^{\frac{5}{6}}\right)^{\frac{3}{4}}$$

$$\ll z^{\frac{1}{4}} (M^{\rho}N^{\sigma})^{\frac{5}{4}} \ll z^{\frac{3}{2}} t^{-\frac{5}{4}} = x^{\frac{1}{4}} t^{-\frac{5}{4}} ,$$

$$F^{-\frac{1}{2}} MN = z^{-\frac{1}{2}} (M^{\rho}N^{\sigma})^{\frac{1}{2}} MN \leq z^{-\frac{1}{2}} (M^{\rho}N^{\sigma})^{\frac{1}{2}} (M^{\frac{1}{3}} N^{\frac{1}{2}})^{3}$$

$$\ll z^{-\frac{1}{2}} (z t^{-1})^{\frac{7}{2}} = x^{\frac{1}{2}} t^{-\frac{7}{2}} ,$$

and

$$(F^{92} M^{171} N^{263})^{\frac{1}{342}} = \left\{z^{92} (M^{\rho}N^{\sigma})^{-92} (M^{\frac{1}{3}} N^{\frac{1}{2}})^{513} (N^{\frac{5}{6}})^{\frac{39}{5}} \right\}^{\frac{1}{342}}$$

$$\ll \left\{z^{92} (z t^{-1})^{\frac{2144}{5}} \right\}^{\frac{1}{342}}$$

$$= x^{\frac{217}{855}} t^{-\frac{1072}{855}} .$$

Since $x^{\frac{1}{4}} t^{-\frac{5}{4}} \leq x^{\frac{217}{855}} t^{-\frac{1072}{855}}$ if and only if $t \leq x$ it follows that

$$S_{\rho,\sigma}(z,t;M,N) \ll x^{\frac{1}{2}} t^{-\frac{7}{2}} + x^{\frac{217}{855}} t^{-\frac{1072}{855}} ,$$

and therefore we have

$$U_1(x,t) + U_2(x,t) \ll \left(x^{\frac{1}{2}} t^{-\frac{7}{2}} + x^{\frac{217}{855}} t^{-\frac{1072}{855}} \right) \log^2 x .$$

Lemma 3.1 now follows from (3.15).

5.4 PROOF OF THEOREM 2.

We first prove a lemma which requires the notion of uniform distributions in r-dimensions.

Lemma 4.1. *Let* $1 < \beta_1 < \ldots < \beta_r$ *be such that* $1, 1/\beta_1, \ldots, 1/\beta_r$ *are linearly independent over* Q. *Let* $N > 1$, *and*

$$A_{\beta_i} = \left\{ \left[a\beta_i \right] : a = 1,2,\ldots, \left[\frac{N+1}{\beta_i} \right] \right\} , \qquad 1 \leq i \leq r .$$

Then, as $N \to \infty$,

$$|A_{\beta_1} \cap \ldots \cap A_{\beta_r}| \sim \frac{N}{\beta_1 \ldots \beta_r} .$$

Moreover, if $r = 1$, *then as* $N \to \infty$,

$$|A_{\beta_1}| = \frac{N}{\beta_1} + O(1)$$

uniformly in β_1 .

Proof. Suppose that $n \in A_{\beta_i}$ $(1 \leq i \leq r)$. Then there exists an integer a_i such that $\left[a_i \beta_i \right] = n$, or $o < a_i \beta_i - n < 1$, or $o < a_i - n/\beta_i < 1/\beta_i$, or

$$1 - \frac{1}{\beta_i} < \frac{n}{\beta_i} - (a_i - 1) < 1.$$

This means that $n \in A_{\beta_1} \cap \ldots \cap A_{\beta_r}$ if and only if the r-dimensional point

$$P_n = \left(\left(\frac{n}{\beta_1}\right), \ldots, \left(\frac{n}{\beta_1}\right)\right),$$

where (N/β_i) denotes the fractional part of n/β_i, lies inside the r-dimensional interval

$$\left\{(x_1, \ldots, x_r) : 1 - \frac{1}{\beta_i} < x_i < 1, \qquad i = 1,2,\ldots, r\right\} \tag{4.1}$$

which has r-dimensional Lebesgue measure $(\beta_1 \ldots \beta_r)^{-1}$. Since $1, 1/\beta_1, \ldots, 1/\beta_r$ are linearly independent over Q if follows from a well known theorem on uniform distributions (see, for example, Cassels [3] Theorem 1, p.64) that the sequence of points (P_n) are uniformly distributed in the r-dimensional unit cube. Therefore the number of points P_n with $n \leqslant N$ which lie inside the interval (4.1) is asymptotic to $N/\beta_1 \ldots \beta_r$ as $N \to \infty$. This proves that

$$|A_{\beta_1} \cap \ldots \cap A_{\beta_r}| \sim \frac{N}{\beta_1 \ldots \beta_r}, \qquad N \to \infty.$$

Finally, if $r = 1$, then

$$|A_{\beta_1}| = \sum_{\substack{1 \leqslant n \leqslant N \\ n = [a\beta_1]}} 1 = \sum_{a < \frac{N+1}{\beta_1}} 1$$

$$= \left[\frac{N+1}{\beta_1}\right] = \frac{N}{\beta_1} + O(1).$$

This completes the proof of the lemma.

Now let $N \geqslant 1$ and write, for each square-free $b > 1$,

$$A_b = \left\{\left[a\, b^{3/2}\right] : a = 1,2,\ldots \left[\frac{N+1}{b^{3/2}}\right]\right\}.$$

For $r = 0,1,2,\ldots$, we define M_r by

$$M_o = N \ , \qquad M_r = \sum_{1<b_1<\ldots<b_r} |A_{b_1} \cap \ldots \cap A_{b_r}| \ ,$$

the summation is over all square-free b_i. We note that A_{b_r} is empty if $b_r^{3/2} > N+1$ so that, corresponding to and fixed N, there are only finitely many positive M_r. Let B_o be the set of positive integers $n \leq N$ which are not in any of the sets A_b. We note that $n \in B_o$ if and only if $n \leq N$ and $f(n) = 0$, where $f(n)$ is defined in our theorem, so that

$$|B_o| = |\{n : n \leq N, \quad n \in F_o\}| \ . \tag{4.2}$$

We also have, from Sylvester's inclusion-exclusion principle, that

$$|B_o| = M_o - M_1 + M_2 - M_3 + M_4 - \ldots \ .$$

Following Brun's observation in the sieve of Eratosthenes we now use a similar device to obtain a quantitative result from this formula. Let H be a positive constant which we shall specify later, and write

$$M_r(H) = \sum_{1<b_1<\ldots<b_r\leq H} |A_{b_1} \ \ldots \ A_{b_r}|, \qquad r = 1,2,\ldots \ .$$

For any <u>even</u> positive integer ℓ we have that

$$|B_o| \leq M_o - M_1(H) + M_2(H) - \ldots + M_\ell(H) \tag{4.3}$$

and

$$|B_o| \geq M_o - M_1 + M_2(H) - \ldots - M_{\ell+1}(H) \ . \tag{4.4}$$

From Lemma 4.1 with $\beta_i = b_i^{3/2}$ we have that

$$|A_{b_1} \cap \ldots \cap A_{b_r}| \sim \frac{N}{(b_1 \ldots b_r)^{3/2}}, \qquad N \to \infty,$$

and so

$$M_r(H) = N \sum_{1<b_1<\ldots<b_1\leqslant H} \frac{\mu^2(b_1)\ldots\mu^2(b_r)}{(b_1\ldots b_r)^{3/2}} + o(N)$$

$$= N\left\{c_r - \delta_r(H)\right\} + o(N)$$

where c_r is defined in the theorem,

$$\delta_r(H) = \sum_{\substack{1<b_1<\ldots<b_r \\ b_r>H}} \frac{\mu^2(b_1)\ldots\mu^2(b_r)}{(b_1\ldots b_r)^{3/2}},$$

and the implied constant depends on r and H. We remark that, for $r \geqslant 1$,

$$0 \leqslant \delta_r(H) \leqslant \left(\sum_{b=2}^{\infty} b^{-3/2}\right)^{r-1} \sum_{b>H} b^{-3/2} \leqslant 2\left(\zeta\left(\frac{3}{2}\right) - 1\right)^{r-1} H^{-\frac{1}{2}}.$$

We also have, from Lemma 4.1 together with the observation that A_b is empty if $b > N^{2/3} + 1$, that

$$M_1 = \sum_{1<b_1\leqslant N^{2/3}} \mu^2(b_1)\left\{\frac{N}{b_1^{3/2}} + O(1)\right\}$$

$$= N\left\{c_1 + O\left(\sum_{b>N^{2/3}} b^{-\frac{3}{2}}\right)\right\} + O(N^{2/3})$$

$$= c_1 N + O(N^{2/3})$$

where the implied constant is absolute.

From (4.3) and (4.4) we now have that

$$\frac{|B_o|}{N} \leqslant c_o - c_1 + \ldots + c_\ell + \delta_1(H) + \delta_3(H) + \ldots + \delta_{\ell-1}(H) + o(1) , \qquad (4.5)$$

and

$$\frac{|B_o|}{N} \geqslant c_o - c_1 + \ldots - c_{\ell+1} - \delta_2(H) - \delta_4(H) - \ldots - \delta_\ell(H) + o(1) , \qquad (4.6)$$

where the implied constants depend on ℓ and H.

As we shall see in the next section, c_r satisfies

$$0 < c_r < \frac{1}{r} c_1 c_{r-1} , \qquad r = 2,3,\ldots$$

so that the series $c_o - c_1 + c_2 - \ldots$ converges to d_o. Let $\varepsilon > 0$. We can choose ℓ so that

$$d_o - \frac{\varepsilon}{4} < c_o - c_1 + \ldots - c_{\ell+1} < c_o - c_1 + \ldots + c_\ell < d_o + \frac{\varepsilon}{4} ,$$

and then choose H so that

$$\delta_1(H) + \delta_2(H) + \ldots + \delta_\ell(H) < \frac{\varepsilon}{4} .$$

It follows from (4.5) and (4.6) that for all sufficiently large N, we have

$$d_o - \varepsilon < \frac{|B_o|}{N} < d_o + \varepsilon .$$

From (4.2) we see that the set F_o has asymptotic density d_o.

The case F_m $(m \geqslant 1)$ can be proved similarly by using the generalised inclusion-exclusion principle (see, for example, [28] Theorem 1, p. 18.), namely

$$|B_m| = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(m+\ell)!}{m!\ \ell!} M_{m+\ell}$$

where B_m is the set of those positive integers $n \leq N$ which lie in exactly m of the sets A_b.

5.5 NUMERICAL VALUES FOR d_m.

We first show that each of the constants c_r in Theorem 2 can be expressed in terms of $\zeta(3a/2)$, $a = 1,2,\ldots,2r$. For $r = 1,2,\ldots$ and $a = 0,1,\ldots$, we define

$$g(r,a) = \sum_{1<b_1<\ldots<b_r} \frac{\mu^2(b_1)\ldots\mu^2(b_r)}{(b_1\ldots b_r)^{3/2}} \sum_{\ell=1}^{r} \frac{1}{b_\ell^{3a/2}} .$$

We note that

$$g(r,0) = r\, c_r , \qquad r = 1,2,\ldots , \tag{5.1}$$

and

$$g(1,a) = \sum_{b=2}^{\infty} \frac{\mu^2(b)}{b^{(3a+3)/2}}$$

$$= \frac{\zeta\left(\dfrac{3a+3}{2}\right)}{\zeta(3a+3)} - 1 , \qquad a = 0,1,\ldots . \tag{5.2}$$

<u>Lemma 5.1.</u> <u>We have</u>, <u>for</u> $r \geq 1$ <u>and</u> $a \geq 0$,

$$g(r+1,a) = g(1,a)c_r - g(r,a+1).$$

<u>Proof.</u> For convenience we write $s = 3/2$ and we denote by $\sum_{(r)}$ the summation with respect to square-free integers $b_1, b_2,\ldots, b_r$ satisfying $1 < b_1<\ldots< b_r$.

We consider

$$\sum_{(r+1)} \frac{1}{(b_1\ldots b_{r+1})^s} \cdot \frac{1}{b_{r+1}^{as}} = \sum_{(r)} \frac{1}{(b_1\ldots b_r)^s} \sum_{b>b_r} \frac{\mu^2(b)}{b^{(a+1)s}} .$$

With $b_o = 1$ we see that

$$\sum_{b>b_r} \frac{\mu^2(b)}{b^{(a+1)s}} = \sum_{b=2}^{\infty} \frac{\mu^2(b)}{b^{(a+1)s}} - \sum_{\ell=1}^{r} \sum_{b_{\ell-1}<b\leq b_\ell} \frac{\mu^2(b)}{b^{(a+1)s}}$$

$$= g(1,a) - \sum_{\ell=1}^{r} \sum_{b_{\ell-1}<b<b_\ell} \frac{\mu^2(b)}{b^{(a+1)s}} - \sum_{\ell=1}^{r} \frac{1}{b_\ell^{(a+1)s}} ,$$

and so

$$\sum_{(r+1)} \frac{1}{(b_1 \dots b_{r+1})^s} \cdot \frac{1}{b_{r+1}^{as}} + \sum_{(r)} \frac{1}{(b_1 \dots b_r)^s} \sum_{\ell=1}^{r} \sum_{b_{\ell-1}<b<b} \frac{\mu^2(b)}{b^{(a+1)s}}$$

$$= \sum_{(r)} \frac{1}{(b_1 \dots b_r)^s} \left\{ g(1,a) - \sum_{\ell=1}^{r} \frac{1}{b_\ell^{(a+1)s}} \right\} ,$$

that is

$$\sum_{(r+1)} \frac{1}{(b_1 \dots b_{r+1})^s} \left\{ \frac{1}{b_{r+1}^{as}} + \sum_{\ell=1}^{r} \frac{1}{b^{as}} \right\} = c_r\, g(1,a) - g(r,a+1) ,$$

which is the required result.

From the reduction formula in Lemma 5.1 we can express each $g(r,a)$ in terms of $g(1,a')$ which, by (5.2), can be calculated from a table of values for the Riemann zeta function. We give the following table of values for $g(r,a)$ truncated to 4 decimal places.

a \ r	1	2	3	4	5	6	7
0	1.1732	1.1949	0.5405	0.1475	0.0277	0.0038	0.0004
1	0.1815	0.1604	0.0638	0.0155	0.0026	0.0003	
2	0.0525	0.0446	0.0171	0.0040	0.0006		
3	0.0170	0.0142	0.0054	0.0012	0.0002		
4	0.0057	0.0047	0.0018	0.0004			
5	0.0020	0.0016	0.0006	0.0001			
6	0.0007	0.0005	0.0002				
7	0.0002	0.0002					

TABLE 1.

From (5.1) we can now calculate c_r giving

$$\left.\begin{array}{lll} c_1 = 1.1732\ldots, & c_2 = 0.5974\ldots, & c_3 = 0.1801\ldots \\ c_4 = 0.0368\ldots, & c_5 = 0.0055\ldots, & c_6 = 0.0006\ldots \end{array}\right\} \quad (5.3)$$

From the formula

$$d_m = \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{(m+\ell)!}{m!\,\ell!} c_{m+\ell}$$

we can now calculate d_m, giving

$$\left.\begin{array}{lll} d_o = 0.275965\ldots, & d_1 = 0.395565\ldots, & d_2 = 0.231299\ldots \\ d_3 = 0.077074\ldots, & d_4 = 0.017015\ldots, & d_5 = 0.002714\ldots \\ d_6 = 0.000331\ldots, & d_7 = 0.000031\ldots, & d_8 = 0.000002\ldots\ . \end{array}\right\} \quad (5.4)$$

We remark that, in order to give allowance for the coefficient of $c_{m+\ell}$ in the formula for d_m, we need more accurate results for c_r than (5.3) to arrive at (5.4); we actually calculated g(r,a) to 8 decimal places. From (5.4) we see that

$$0 < 1 - \sum_{m=0}^{8} d_m < 0.000004 ,$$

and

$$0 < c_1 - \sum_{m=0}^{8} md_m < 0.00002 .$$

In order to verify (5.4) we use a computer to find the following empirical frequencies for f(n) in $1 \leq n \leq N$ when $N = 5000$.

m	0	1	2	3	4	5	6
B_m	1485	2049	1087	313	58	7	1
d'_m	0.2970	0.4098	0.2174	0.0626	0.0116	0.0014	0.0002

TABLE 2.

The table here shows some small agreement with out predicted frequencies and we note in particular that

$$d'_o > d_o , \quad d'_m < d_m \qquad m = \;\; 2, \ldots \; .$$

$d'_1 > d_1$,

We give the following explanation for this. Let us put

$$R(x) = \frac{Q(x) - [\sqrt{x}]}{[\sqrt{x}]} ,$$

the ratio of the number of square-full integers which are not squares to the number of squares. We have

$$\lim_{x\to\infty} R(x) = A_{22} - 1 = c_1 = 1.1732\ldots,$$

by the asymptotic formula (1.2). From Table 2 we see that $Q(25 \times 10^6) = 10435$ giving

$$R(25 \times 10^6) = 1.087$$

which is substantially smaller than its limiting value c_1. This means that, up to 25 million, we are still in the initial block of the sequence (q_n) where the squares show up more frequently than it should asymptotically. That is the second dominating term, namely $A_{23}\, x^{1/3}$ where $A_{23} < 0$, still interferes with the result; indeed up to 10^5, the squares actually form a majority in the sense that $R(10^5) < 1$. We therefore expect d'_o to decrease while d'_m $(m \geqslant 1)$ to increase to our theoretical values as N increases.

We remark that our computation also show that

$$|\Delta_2(x)| \leqslant 3 \quad \text{for} \quad 1 \leqslant x \leqslant 25 \times 10^6 .$$

We also mention that the least solution to $f(n) = 6$ is $n = 3611$, and the 6 square-full integers in the interval $3611^2 < q < 3612^2$ are

$$\begin{aligned}
13{,}041{,}125 &= 323^2.5^3\\
13{,}041{,}675 &= 695^2.3^3\\
13{,}042{,}575 &= 195^2.7^3\\
13{,}043{,}800 &= 35^2.22^3\\
13{,}045{,}131 &= 99^2.11^3\\
13{,}048{,}832 &= 1277^2.2^3 \quad .
\end{aligned}$$

5.6 PROOF OF (1.8).

We first show that, as $n \to \infty$,

$$q_n = a\, n^2 + b\, n^{5/3} + O(n^{4/3}) \tag{6.1}$$

where

$$a = \frac{1}{A_{22}^2}, \qquad b = \frac{2A_{23}}{A_{22}^{2/3}}. \tag{6.2}$$

We shall use the result

$$\Delta_2(x) \ll x^{\frac{1}{6}}, \qquad x \to \infty$$

together with the observation that $Q(q_n) = n$. From (1.2) we have, as $n \to \infty$,

$$n = A_{22}\, q_n^{\frac{1}{2}} + A_{23}\, q_n^{\frac{1}{3}} + O(q_n^{\frac{1}{6}})$$

and so

$$n^2 = A_{22}^2\, q_n + 2A_{22}\, A_{23}\, q_n^{\frac{5}{6}} + O(q_n^{\frac{2}{3}}). \tag{6.3}$$

Since $q_n \ll n^2$ we now have, from (6.2) and (6.3),

$$q_n = a\, n^2 + O(n^{\frac{5}{3}})$$

and so

$$q_n^{\frac{5}{6}} = a^{\frac{5}{6}}\, n^{\frac{5}{3}} + O(n^{\frac{4}{3}}).$$

The required result (6.1) now follows from (6.3) and (6.2).

Now given any n, there exists a unique positive integer m such that

$$m^2 \leqslant q_n < q_{n+1} \leqslant (m+1)^2$$

and so

$$q_{n+1} - q_n \leq 2m+1 \quad . \tag{6.4}$$

From (6.1) we see that, as $n \to \infty$,

$$m^2 = a\, n^2 + O(n^{\frac{5}{3}})$$

and so

$$m = \sqrt{a}\, n + O(n^{\frac{2}{3}}) \quad .$$

From (6.4) we now have

$$\frac{q_{n+1} - q_n}{2n} \leq \sqrt{a} + O(n^{-\frac{1}{3}})$$

and hence

$$\limsup_{n\to\infty} \frac{q_{n+1} - q_n}{2n} \leq \frac{1}{A_{22}}$$

by (6.2). Finally, since $d_o > 0$, there are infinitely many n such that $q_n = m^2$ and $q_{n+1} = (m+1)^2$, so that (6.4) holds with equality infinitely often, and so the required result (1.8) follows.

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